HW 3a Solutions

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1 The camera is at (0, 0, 0), facing in the $-z$ direction, with the $+y$ direction up.

1.1 What are the camera's basis vectors?

Note: There are two valid answers for $\hat{w}$. Which one you use depends on a number of factors. Old graphics resources tend to set $\hat{w}$ in the direction of the gaze. This results in a left-handed basis. Newer graphics resources will use more mathematically correct notation and set $\hat{w}$ pointing against the gaze, but resulting in a right-handed basis. I will be using the left-handed basis (with $\hat{w}$ facing away from the gaze). If you used the right-handed basis in your answers, replace $\hat{w}$ with $-\hat{w}$ throughout the solution.

$$\hat{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \hat{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \hat{w} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \text{ (LHB), or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ (RHB)}$$

1.2 What are the rays for the pixels (1, 1) and (0, 0)?

The vector $\vec{d}$ through the appropriate pixels should be of the form $\alpha\hat{u} + \beta\hat{v} + \hat{w}$.

The equations for α and β are nearly identical, just switch out x and y related values where appropriate. α is composed of two factors - one to adjust for the field of view, and one to adjust for what x value the associated pixel will have. Here, I will use θ as the field of view angle.

$$\begin{aligned} \alpha &= \tan\left(\frac{\theta}{2}\right) \frac{2x - width + 1}{width} \\ \beta &= \tan\left(\frac{\theta}{2}\right) \frac{2y - height + 1}{height} \end{aligned}$$

By doubling x, then subtracting off the width, we are shifting the range of x from $[0, width - 1]$ to $[1 - width, width - 1]$. We will divide this by the width to keep the range to $[-1, 1]$. We must take the tangent of $\frac{\theta}{2}$ to account for the field of view. In this case, $\theta = 90$, so the tangent works out to be 1.

$$
\begin{aligned}
\vec{d} \text{ at } (1,1) &= \frac{2(1)-3+1}{3}\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{2(1)-3+1}{3}\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\-1\end{bmatrix} \\
f(t) \text{ at } (1,1) &= \begin{bmatrix}0\\0\\0\end{bmatrix} + t\vec{d} = t\begin{bmatrix}0\\0\\-1\end{bmatrix} \\
\vec{d} \text{ at } (0,0) &= \frac{2(0)-3+1}{3}\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{2(0)-3+1}{3}\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\-1\end{bmatrix} = \begin{bmatrix}\frac{-2}{3}\\\frac{-2}{3}\\-1\end{bmatrix} \\
f(t) \text{ at } (1,1) &= \begin{bmatrix}0\\0\\0\end{bmatrix} + t\vec{d} = t\begin{bmatrix}\frac{-2}{3}\\\frac{-2}{3}\\-1\end{bmatrix}
\end{aligned}
$$

1.3 What is the ray for the lower-left corner of the field of view?

The equation for this should look the same as above, but we will leave out the term of α and β designed to adjust for a pixel and substitute in -1 (since we are looking at the bottom left corner).

$$
\begin{aligned}
\vec{d} &= -1\begin{bmatrix}1\\0\\0\end{bmatrix} + -1\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\end{bmatrix} \\
f(t) &= \begin{bmatrix}0\\0\\0\end{bmatrix} + t\vec{d} = t\begin{bmatrix}-1\\-1\\-1\end{bmatrix}
\end{aligned}
$$

2 The eye point is at (0, 5, 5), the target point is at (0, 0, 0), and the up vector is (0, 1, 0).

2.1 What are the camera's basis vectors?

The $\hat{w}$ vector is the vector looking along the direction of view, so we can get $\hat{w}$ by subtracting the eye point from the target point, then normalizing.

We can obtain a $\hat{u}$ vector by crossing $\hat{w}$ with the up vector, then renormalizing. (Note: if you are using the RHB, $\hat{u}$ can be obtained by crossing 'up' with $\hat{w}$. Both methods of finding $\hat{u}$ should yield the same value though.)

Similarly, we can obtain a $\hat{v}$ vector by crossing $\hat{u}$ with $\hat{w}$. Note that we won't have to renormalize here.

$$\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -5 \end{bmatrix}$$

$$\hat{w} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$\vec{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

2.2 What are the rays for the pixels (1, 1) and (0, 0)?

The same equations from above apply, we just simply substitute in our new $\hat{u}$, $\hat{v}$, and $\hat{w}$ vectors.

$$\begin{aligned}
\vec{d} \text{ at } (1,1) &= \frac{2(1)-3+1}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{2(1)-3+1}{3} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \\
f(t) \text{ at } (1,1) &= \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} + t\vec{d} = \begin{bmatrix} 0 \\ 5 - \frac{t}{\sqrt{2}} \\ 5 - \frac{t}{\sqrt{2}} \end{bmatrix} \\
\vec{d} \text{ at } (0,0) &= \frac{2(0)-3+1}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{2(0)-3+1}{3} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{3} \\ \frac{-\sqrt{2}}{3} - \frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{3} - \frac{1}{\sqrt{2}} \end{bmatrix} \\
f(t) \text{ at } (0,0) &= \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} + t\vec{d} = \begin{bmatrix} \frac{-2t}{3} \\ 5 - \frac{t\sqrt{2}}{3} - \frac{t}{\sqrt{2}} \\ 5 + \frac{t\sqrt{2}}{3} - \frac{t}{\sqrt{2}} \end{bmatrix}
\end{aligned}$$

2.3 What is the ray for the top-center of the field of view?

The ray for the top center of the field of view is the same as the above equations again, but using 0 as the factor for $\hat{u}$, and 1 as the factor for $\hat{v}$. Please remember that we are omitting the field of view factor for the sake of simplicity, we have already shown that it is simply 1.

$$
\begin{aligned}
\vec{d} &= 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\sqrt{2} \end{bmatrix} \\
f(t) &= \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} + t\vec{d} = \begin{bmatrix} 0 \\ 5 \\ 5 - t\sqrt{2} \end{bmatrix}
\end{aligned}
$$

3 The camera is at point $\vec{o}$ and its basis is $\{\hat{u}, \hat{v}, \hat{w}\}$.

3.1 What is the ray for the pixel (1, 1)?

$$
\begin{aligned}
\vec{d} \text{ at } (1,1) &= \frac{2(1) - 3 + 1}{3}\hat{u} + \frac{2(1) - 3 + 1}{3}\hat{v} + \hat{w} = \hat{w} \\
f(t) \text{ at } (1,1) &= \vec{o} + t\vec{d} = \vec{o} + t\hat{w}
\end{aligned}
$$

3.2 What is the ray for the pixel (0, 0)?

$$
\begin{aligned}
\vec{d} \text{ at } (0,0) &= \frac{2(0) - 3 + 1}{3}\hat{u} + \frac{2(0) - 3 + 1}{3}\hat{v} + \hat{w} \\
&= \frac{-2}{3}\hat{u} + \frac{-2}{3}\hat{v} + \hat{w} \\
f(t) \text{ at } (0,0) &= \vec{o} + t\vec{d} = \vec{o} + \frac{-2t}{3}\hat{u} + \frac{-2t}{3}\hat{v} + t\hat{w}
\end{aligned}
$$