# HW 2a Solutions

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# 1 Convolution Filters

## 1.1 Convolution Properties

**Prove the following properties about convolution.** The general way to prove these properties is to assume the property is true, and to set the involved quantities equal to each other. I then expand them out via their definition and manipulate the left hand side so that it becomes equal to the right hand side.

1.

$$\begin{aligned} (f \star g)[j] &= (g \star f)[j] \\ \sum_i f[i]g[j-i] &= \sum_i g[i]f[j-i] \\ \text{let } i &= j-k \\ \sum_i f[i]g[j-i] &= \sum_k g[j-k]f[k] \end{aligned}$$

2.

$$\begin{aligned} (f \star b)[j] &= f[j] \\ \sum_i f[i]b[j-i] &= f[j] \\ f[1]b[j-1] + \ldots + f[j]b[j-j] + \ldots + f[n]b[j-n] &= f[j] \\ f[1] * 0 + \ldots + f[j] * 1 + \ldots + f[1] * 0 &= f[j] \\ f[j] &= f[j] \end{aligned}$$

3.

$$\begin{aligned}
((\alpha f) \star g)[j] &= \alpha (f \star g)[j] \\
\sum_i (\alpha f[i]) g[j-i] &= \alpha \sum_i f[i]g[j-i] \\
\alpha \sum_i f[i]g[j-i] &= \alpha \sum_i f[i]g[j-i]
\end{aligned}$$

4.

$$\begin{aligned}
(f \star (g+h))[j] &= (f \star g)[j] + (f \star h)[j] \\
\sum_i f[i] (g[j-i] + h[j-i]) &= \sum_i f[i]g[j-i] + \sum_i f[i]h[j-i] \\
\sum_i f[i]g[j-i] + f[i]h[j-i] &= \sum_i f[i]g[j-i] + \sum_i f[i]h[j-i] \\
\sum_i f[i]g[j-i] + \sum_i f[i]h[j-i] &= \sum_i f[i]g[j-i] + \sum_i f[i]h[j-i]
\end{aligned}$$

5.

$$\begin{aligned}
((f \star g) \star h)[j] &= (f \star (g \star h))[j] \\
\sum_{k_1} \left( \sum_{i_1} f[i_1]g[k_1 - i_1] \right) h[j-k_1] &= \sum_{k_2} f[k_2] \left( \sum_{i_2} g[i_2]h[j-k_2-i_2] \right) \\
\sum_{i_1} f[i_1] \left( \sum_{k_1} g[k_1 - i_1]h[j-k_1] \right) &= \sum_{k_2} f[k_2] \left( \sum_{i_2} g[i_2]h[j-k_2-i_2] \right) \\
\text{let } k_1^* &= k_1 - i_1 \\
\sum_{i_1} f[i_1] \left( \sum_{k_1^*} g[k_1^*]h[j-i_1-k_1^*] \right) &= \sum_{k_2} f[k_2] \left( \sum_{i_2} g[i_2]h[j-k_2-i_2] \right)
\end{aligned}$$

## 1.2 Unsharp Mask

**Use these properties to derive a discrete convolution filter that is equivalent to the "unsharp mask".**

- Begin with the image $I_{\text{in}}$.
- Blur $I$ with a gaussian filter of width $\sigma = 2$ pixels, storing the result in $I_{\text{blur}}$.
- Set the final image $I_{\text{out}}$ to $(1+\beta)I_{\text{in}} - \beta I_{\text{blur}}$.

In the equations below, I will represent the gaussian filter with $H$, and the identity filter with $B$.

$$
\begin{aligned}
I_{out}[x,y] &= (1+\beta)\, I_{in}[x,y] - \beta\,(H \star I_{in})\,[x,y] \\
I_{out}[x,y] &= (1+\beta)\,(B \star I_{in})\,[x,y] - \beta\,(H \star I_{in})\,[x,y] \\
I_{out}[x,y] &= (((1+\beta)\,B) \star I_{in})\,[x,y] + ((-\beta H) \star I_{in})\,[x,y] \\
I_{out}[x,y] &= (((1+\beta)\,B - \beta H) \star I_{in})\,[x,y] \\
&\quad \text{Let } UMS = (1+\beta)\,B - \beta H \\
I_{out}[x,y] &= (UMS \star I_{in})\,[x,y]
\end{aligned}
$$

So now we have our single filter - $UMS$. Let's write $UMS$ as a function of its inputs $t$ and $s$. It is important to mention that UMS is defined as being centered at (0, 0), and spreading out over some two dimensional radius r (5 in this case). Thus, UMS covers $(2r+1)(2r+1) = 121$ values in our case.

$$
UMS[t,s] = \begin{cases} 1 + \beta\frac{8\pi - 1}{8\pi} & \text{if } t, s = 0 \\ -\frac{\beta}{8\pi} e^{-\frac{t^2+s^2}{16}} & \text{otherwise} \end{cases}
$$

Since this filter is based off of the gaussian filter from the class notes, we know it is separable, and that each of the new functions are identical. What's more, each of the 1 dimensional functions are symmetric. Thus, we only need to know 6 values (the center, plus each value along the 5 unit radius) to define the filter. Upon inspection, it becomes clear that the a value at $H_x(\pm n) = H_y(\pm n) = \sqrt{H(n,n)}$. I will not go into why, but for a simple example, look at the lecture slides. So, our list looks like:

$$H_x = \beta * \{0.4461, 0.3937, 0.2709, 0.1449, 0.0605, 0.1962\}$$

The final thing to mention is the special value at $(0,0)$, which is $\approx 1 + .96\beta$.

# 2 Reconstruction and Resampling

## 2.1 Filter Efficiency

**This code is inefficient because it loops over the whole array. If we know that the filter radius is $r$ (that is $h(t) = 0$ for $|t| > r$), what should the loop bounds be to make the minimum number of computations while still computing the correct result? Assume you have the functions `round`, `floor`, and `ceil` available.**

Since there is no point in evaluating the filter function when $|t| > r$, then the lower bound, $i_{min}$ is found by $x - r \geq i_{min} \Rightarrow i_{min} = \texttt{ceil}(x - r)$ and the upper bound, $i_{max}$ is found by $x + r \leq i_{max} \Rightarrow i_{max} = \texttt{floor}(x + r)$.

## 2.2 Ripple-Free and Interpolating Filters

**A reconstruction filter is *interpolating* if $g(x) = f(x)$ for the original sample points (that is, when $x$ is an integer). It is *ripple-free* if $g$ is a constant function whenever $f$ is a constant sequence (when $f[i]$ has the same value for all $i$). Give criteria that one can use to examine a given filter $h$ and determine whether each of these properties holds.**

In order for $g(x) = f(x)$ for all original sample points, then the filter $h$ needs to be equal to zero at all integer points except at 0, where it equals 1. It will look like the identity filter at integer points.

$$h(x) = \begin{cases} 0 & x \neq 0, x \in \mathbb{Z} \\ 1 & x = 0 \end{cases}$$

In order for $g(x)$ to be ripple-free, we must be in a situation where $f(x) = c$ and we want $g(x) = c$ too. Thus:

$$\begin{aligned} g(x) &= \sum_i f[i]h(x-i) \ \forall x \\ c &= \sum_i ch(x-i) \ \forall x \\ 1 &= \sum_i h(x-i) \ \forall x \end{aligned}$$

This last equation must be satisfied for $h$ to be ripple free.

# 3 Math Review

## 3.1 Inverse Functions

**Which of the following functions has an inverse?**

(a) cannot have an inverse - there are many elements in the preimage of (a) that map to any single element in the image of (a). This cannot deterministically be undone, hence (a) cannot have an inverse. (c) cannot have an inverse either - any valid element of the domain does not necessarily map back to the range. There is no $x$ such that $f(x) = 2$ for example (remember that we are only allowed to use integers for this function!). (b) does not map every element in $\mathbb{R}$ to a unique element in $\mathbb{R}$. This is easy to see because there are two points where $x^3 + \frac{x^2}{100} = 0$ - $x = 0$ and $x = \frac{-1}{100}$. Thus, no function had an inverse.

## 3.2 Function Properties

**Consider the function** $f\colon \mathbb{R} \to \mathbb{R}^2\colon f(t) = (\cos t, \sin t)$

### 3.2.1 What is the range of $f$?

The range is the set of valid values $f$ can evaluate to. In this case, the range of $f$ is the unit circle in $\mathbb{R}^2$. It is *not* $[-1, 1] \times [-1, 1]$, because this would imply that $f$ could map to any element on the unit square in each quadrant. For example, $f$ cannot map to $(0, 0)$, which is clearly in $[-1, 1] \times [-1, 1]$.

### 3.2.2 What is the image of the interval $[0, \pi/2]$ under $f$?

The image of a set of values under a function is the set of values that the function produces after receiving the input set. This input set, by the way, is called the preimage. In this case, our image is $\{(x, y)\,|x^2 + y^2 = 1, x \geq 0, y \geq 0\}$. The reason we can define our image like this, is because the values of x and y are correlated. In particular, they are bound by the pythagorean theorem, on a triangle with a hypotenuse of length 1. Its easy to see that this is the quarter of the unit circle that lies in the first quadrant. The above notation is just that - formal notation to describe the image.

### 3.2.3 What is the preimage of the set $\{(x, y)\,||x| < \sqrt{2}\}$?

See above for definition of preimage. $\bigcup_{n \in \mathbb{Z}} \left(4\pi n + \frac{\pi}{4}, 4\pi n + \frac{3\pi}{4}\right)$.

## 3.3 Orthonormal Basis

**Given the non-parallel 3D vectors $a$ and $b$, compute a right-handed orthonormal basis such that $u$ is parallel to $a$, and $w$ is normal to the planes defined by $a$ and $b$.**

First, we'll find $u$ such that it is equal to $a$ in direction, but has a magnitude of 1. Then, we will find $v$ such that it is in the plane defined by $a$ and $b$, but perpendicular to $u$, and has a magnitude of 1. Then, we only need to set $w = u \times v$. In the equations below, I differentiate between a normalized vector and an unnormalized vector by giving the normalized version an hat. There are a number of ways at arriving at a right handed orthonormal basis. Two ways are shown below. Any would do.

$$\begin{aligned} u = a, \hat{u} &= \frac{u}{\|u\|} \\ v = b - (b \cdot a)\, a, \hat{v} &= \frac{v}{\|v\|} \\ \hat{w} &= \hat{u} \times \hat{v} \end{aligned}$$

$$\begin{aligned} \hat{u} &= \frac{a}{\|a\|} \\ w = \hat{u} \times b, \hat{w} &= \frac{w}{\|w\|} \\ \hat{v} &= \hat{w} \times \hat{u} \end{aligned}$$

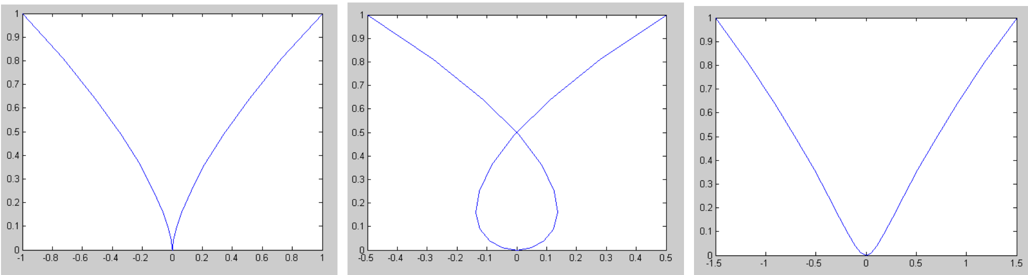

Figure 1: 4b - Left: $a = 0$, Middle: $a = \frac{-1}{2}$, Right: $a = \frac{1}{2}$

# 4 Curves and Surfaces

## 4.1 Implicit Surface

**What is the implicit equation of the plane through the 3D points (1, 0, 0), (0, 1, 0), and (0, 0, 1)? What is the parametric equation? What is the normal vector to this plane?**

Implicit equation: $x + y + z - 1 = 0$. There are many valid solutions.

Parametric equation: $\vec{f}(s, t) = \begin{bmatrix} s \\ t \\ 1 - s - t \end{bmatrix}$. Many solutions here too.

Normal vector: $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Sorry, there's only one normal vector! (Note that we didn't aske for a unit normal vector so being off by a scalar factor is ok.

## 4.2 Parametric Curves

**Consider the parametric curve $x = t^3 + at, y = t^2$. What does the curve look like as $t$ ranges over $[-1, 1]$ when $a = 0$? When $a > 0$? When $a < 0$? Draw rough sketches of the shape.**

Refer to Figure 1.

## 4.3 Parametric Surfaces

**Consider the parametric surface $x = t^3 + st, y = t^2, z = s$. What does the surface look like as $(s, t)$ ranges over $[-1, 1] \times [-1, 1]$? Sketch the intersections of the surface with the three coordinate planes (the planes at $x = 0$, $y = 0$, and $z = 0$).**

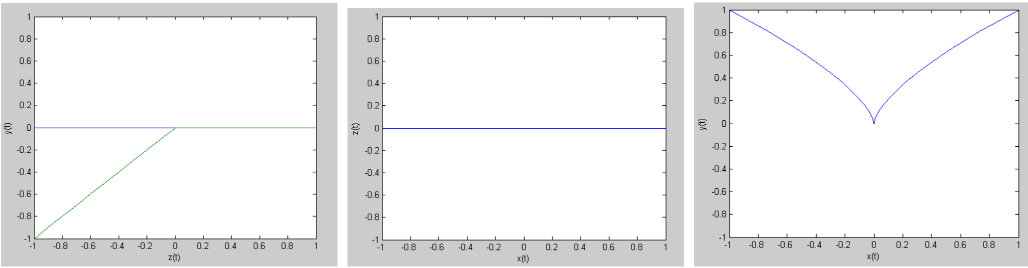

Figure 2: 4c - Left: $x = 0$, Middle: $y = 0$, Right: $z = 0$

Refer to Figure 2 for the $x = 0$, $y = 0$, and $z = 0$ slices of the function. If you were positioned on the +z axis, and marched in the -z direction, you would see slices of this function that looked like the above functions, where your position on the z-axis would be your s value.