CS 465 Homework 2a

(revised September 19, 2003)

out: Monday 15 September 2003
due: Monday 22 September 2003

Problem 1: Convolution filters

Here is a definition for the discrete convolution of two sequences of numbers, f and g:

$$(f \star g)[j] = \sum_i f[i]g[j-i]$$

That is, we can compute the value of the convolution at any index j by summing products of the elements of f and g with an offset that depends on j. In the following let b be the sequence that has $b[0] = 1$ and $b[i] = 0$ for all other i.

This sum runs over all i, so there are no boundary conditions to worry about. You can think of this as saying that out-of-bounds accesses to the arrays f and g will return zero.

1. Verify that discrete convolution has the following properties:

$f \star b = f = b \star f$	b is an identity
$(\alpha f) \star g = \alpha(f \star g)$	scalars factor out
$f \star (g + h) = f \star g + f \star h$	distributes over $+$
$f \star g = g \star f$	commutative
$(f \star g) \star h = f \star (g \star h)$	associative

   Hints: The general approach is to expand out the definitions on both sides and then manipulate one side to show it's equal to the other. Also, remember that you can use a change of variable in a sum: if $p : \mathbb{Z} \to \mathbb{Z}$ is a function that renumbers the terms without losing any or duplicating any (that is, it's a bijection), then

$$\sum_i X(i) = \sum_i X(p(i))$$

   where X can be any expression that depends on i.

2. Use these properties to derive a radius 5 (11 by 11) discrete convolution filter that is equivalent to the "unsharp mask" procedure:

- Begin with the image I_{in}.
- Blur I with a radius 5 gaussian filter of width $\sigma = 2$ pixels, storing the result in I_{blur}.
- Set the final image I_{out} to $(1+\beta)I_{\text{in}} - \beta I_{\text{blur}}$.

List the values of the filter itself (round to 2 significant figures), but take advantage of separability and symmetry to avoing having to write down 121 numbers (fewer than 10 should suffice). Explain where you used the properties from part 1 in your derivation.

For reference, the 2D Gaussian filter with width σ is defined as $h(s,t) = h(s)h(t)$ where $h(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma)^2}$.

Problem 2: Reconstruction and resampling

The equation that defines reconstruction of a continuous function g from a discrete sequence of samples f is

$$g(x) = \sum_i f[i]h(x-i).$$

This can be translated into pseudocode as:

```
g(f, h, x) {
   // Evaluates the reconstruction g of the samples f using the
   // filter h at the point x.
   result = 0;
   for i = 0 to N - 1
      result += f[i] * h(x - i);
   return result;
}
```

where `N` is the number of elements in the array `f`.

1. This code is inefficient because it loops over the whole array. If we know that the filter radius is r (that is, $h(t) = 0$ for $|t| > r$) What should the loop bounds be to make the minimum number of computations while still computing the correct result? Assume you have the functions `round`, `floor`, and `ceil` available.[1]

2. A reconstruction filter is *interpolating* if $g(x) = f(x)$ for the original sample points (that is, when x is an integer). It is *ripple-free* if g is a constant function whenever f is a constant sequence (when $f[i]$ has the same value for all i). Give criteria that one can use to examine a given filter h and determine whether each of these properties holds.

[1] The value round(x) is the nearest integer to x; floor(x) is the greatest integer that is $\leq x$, and ceil(x) is the smallest integer that is $\geq x$.

Again, feel free to assume that out-of-bounds accesses will return zero.

Problem 3: Math review questions

1. Which of the following functions has an inverse?
   (a) $f : \mathbb{R} \to \mathbb{Z} : x \mapsto \mathrm{round}(x)$
   (b) $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^3 + x^2/100$
   (c) $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto x^3 + x^2/100$
2. Consider the function $f : \mathbb{R} \to \mathbb{R}^2 : f(t) = (\cos t, \sin t)$.
   (a) What is the range of f?
   (b) What is the image of the interval $[0, \pi/2]$ under f?
   (c) What is the preimage of the set $\{(x, y) \mid |x| < \sqrt{2}\}$?
3. Show that the cross product is not associative by giving a counterexample using the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.
4. Shirley Exercise 2.9.[2]

Problem 4: Curves and surfaces

1. Shirley Exercise 2.12.
2. Consider the parametric curve

$$x = t^3 + at$$
$$y = t^2.$$

   What does the curve look like as t ranges over from $[-1, 1]$ when $a = 0$? When $a > 0$? When $a < 0$? Draw rough sketches of the shape.
3. Consider the parametric surface

$$x = t^3 + st$$
$$y = t^2$$
$$z = s.$$

   What does the surface look like as (s, t) ranges over $[-1, 1] \times [-1, 1]$? Sketch the intersections of the surface with the three coordinate planes (the planes $x = 0$, $y = 0$, and $z = 0$).

[2] By 2.9 I mean Exercise 9 in Chapter 2.