Cubic Bézier Curves

Abe Davis

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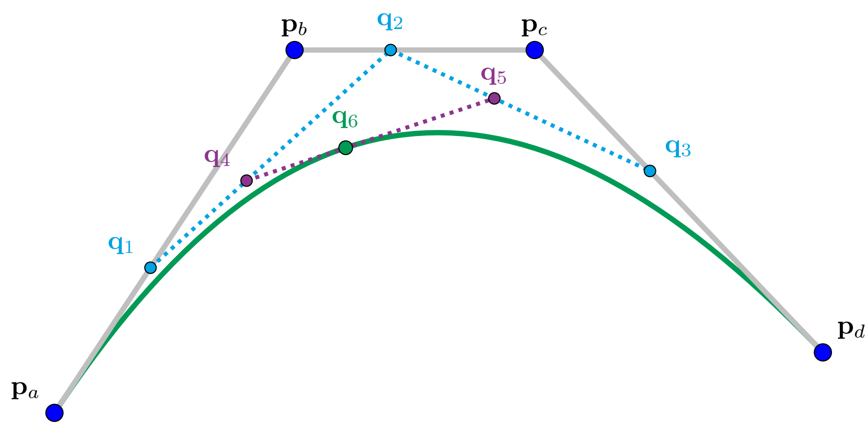

We start with the ordered set of four control points $\mathbf{P} = \{\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c, \mathbf{p}_d\}$. This yields three edges in the cage of our spline. Our first step will be to linearly interpolate along each of these edges by an amount α to find the points $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$:

$$\mathbf{q}_1 = (1-\alpha)\mathbf{p}_a + \alpha\mathbf{p}_b$$
$$\mathbf{q}_2 = (1-\alpha)\mathbf{p}_b + \alpha\mathbf{p}_c$$
$$\mathbf{q}_3 = (1-\alpha)\mathbf{p}_c + \alpha\mathbf{p}_d$$

This gives us two new edges $\overrightarrow{\mathbf{q}_1\mathbf{q}_2}$ and $\overrightarrow{\mathbf{q}_2\mathbf{q}_3}$. We now repeat the process and linearly interpolate by α along these two new edges to find $\{\mathbf{q}_4, \mathbf{q}_5\}$:

$$\mathbf{q}_4 = (1-\alpha)\mathbf{q}_1 + \alpha\mathbf{q}_2$$
$$\mathbf{q}_5 = (1-\alpha)\mathbf{q}_2 + \alpha\mathbf{q}_3$$

This leaves us with one final edge, $\overrightarrow{\mathbf{q}_4\mathbf{q}_5}$. If we linearly interpolate by α along this edge, we find our spline point $\mathbf{q}_6 = \mathbf{s}_{\mathbf{P}}(\alpha)$ linearly interpolate by α:

$$\mathbf{q}_6 = (1-\alpha)\mathbf{q}_4 + \alpha\mathbf{q}_5$$

The first thing you should notice here is that every step of our process just calculated a linear interpolation between consecutive points using the parameters α and $(1-\alpha)$. A linear combination of linear combinations is also a linear combination, so this ensures that our final point, $\mathbf{q}_6$, can be expressed as some linear combination of the original points $\{\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c, \mathbf{p}_d\}$.

Now, let's figure out what that linear combination is.

First, let's replace the terms $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ in our definitions of $\mathbf{q}_4$ and $\mathbf{q}_5$ with their individual definitions given in terms of our original points:

$$\begin{aligned}\mathbf{q}_4 &= (1-\alpha)\left[(1-\alpha)\mathbf{p}_a + \alpha\mathbf{p}_b\right] + \alpha\left[(1-\alpha)\mathbf{p}_b + \alpha\mathbf{p}_c\right] \\ &= (1-\alpha)^2\mathbf{p}_a + 2\alpha(1-\alpha)\mathbf{p}_b + \alpha^2\mathbf{p}_c\end{aligned}$$

$$\begin{aligned}\mathbf{q}_5 &= (1-\alpha)\left[(1-\alpha)\mathbf{p}_b + \alpha\mathbf{p}_c\right] + \alpha\left[(1-\alpha)\mathbf{p}_c + \alpha\mathbf{p}_d\right] \\ &= (1-\alpha)^2\mathbf{p}_b + 2\alpha(1-\alpha)\mathbf{p}_c + \alpha^2\mathbf{p}_d\end{aligned}$$

And now we can substitute these equations into our equation for $\mathbf{q}_6$:

$$\begin{aligned}\mathbf{q}_6 &= (1-\alpha)\left[(1-\alpha)^2\mathbf{p}_a + 2\alpha(1-\alpha)\mathbf{p}_b + \alpha^2\mathbf{p}_c\right] + \alpha\left[(1-\alpha)^2\mathbf{p}_b + 2\alpha(1-\alpha)\mathbf{p}_c + \alpha^2\mathbf{p}_d\right] \\ &= \mathbf{p}_a\left[(1-\alpha)^3\right] + \mathbf{p}_b\left[2\alpha(1-\alpha)^2 + \alpha(1-\alpha)^2\right] + \mathbf{p}_c\left[\alpha^2(1-\alpha) + 2\alpha^2(1-\alpha)\right] + \mathbf{p}_d\left[\alpha^3\right] \\ &= \mathbf{p}_a\left[-\alpha^3 + 3\alpha^2 - 3\alpha + 1\right] + \mathbf{p}_b\left[3\alpha^3 - 6\alpha^2 + 3\alpha\right] + \mathbf{p}_c\left[-3\alpha^3 + 3\alpha^2\right] + \mathbf{p}_d\left[\alpha^3\right]\end{aligned}$$

Which we can write as the dot product

$$\mathbf{q}_6 = \begin{bmatrix} -\alpha^3 + 3\alpha^2 - 3\alpha + 1 \\ 3\alpha^3 - 6\alpha^2 + 3\alpha \\ -3\alpha^3 + 3\alpha^2 \\ \alpha^3 \end{bmatrix}^{\intercal} \begin{bmatrix} \mathbf{p}_a \\ \mathbf{p}_b \\ \mathbf{p}_c \\ \mathbf{p}_d \end{bmatrix}$$

Now, let's separate the coefficients of the vector on our left from the powers of α:

$$\mathbf{q}_6 = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_a \\ \mathbf{p}_b \\ \mathbf{p}_c \\ \mathbf{p}_d \end{bmatrix}$$

We could alternatively write this with our α terms on the right side as:

$$\mathbf{q}_6 = \begin{bmatrix} \mathbf{p}_a & \mathbf{p}_b & \mathbf{p}_c & \mathbf{p}_d \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \\ \alpha^3 \end{bmatrix}$$