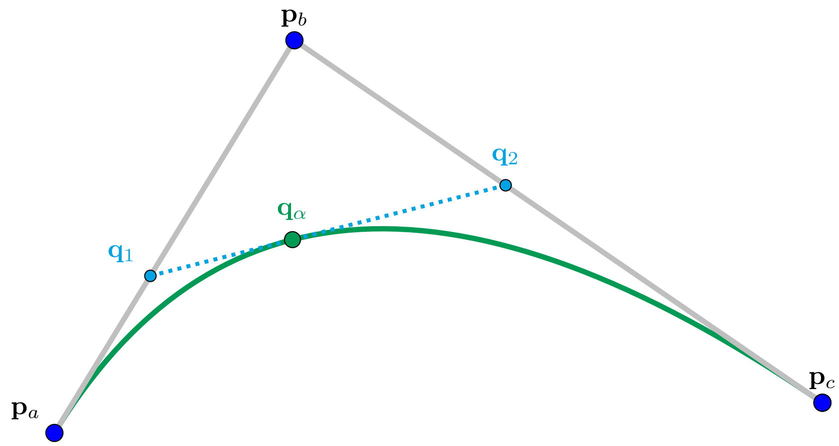
Quadratic Bézier Curves

Abe Davis

September 8, 2023



We start with the ordered set of three control points $\mathbf{P} = \{\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c\}$. This yields two edges in the cage of our spline. Our first step will be to linearly interpolate along each of these edges by an amount α to find the points $\{\mathbf{q}_1, \mathbf{q}_2\}$:

$$\mathbf{q}_1 = (1-\alpha)\mathbf{p}_a + \alpha\mathbf{p}_b$$
$$\mathbf{q}_2 = (1-\alpha)\mathbf{p}_b + \alpha\mathbf{p}_c$$

This gives us one new edge $\overrightarrow{\mathbf{q}_1\mathbf{q}_2}$. We now repeat the process and linearly interpolate by α along this edge to find our spline point, $\mathbf{q}_\alpha$:

$$\mathbf{q}_\alpha = (1-\alpha)\mathbf{q}_1 + \alpha\mathbf{q}_2$$

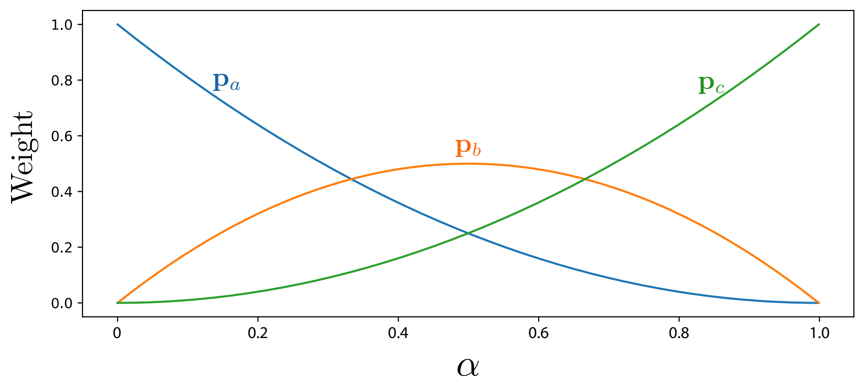
The first thing you should notice here is that every step of our process just calculated a linear interpolation between consecutive points using the parameters α and $(1-\alpha)$. A linear combination of linear combinations is also a linear combination, so this ensures that our final point, $\mathbf{q}_\alpha$, can be expressed as some linear combination of the original points $\{\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c\}$. Now, let's figure out what that linear combination is. To do this, we will simply substitute the equations we derived for $\mathbf{q}_1$ and $\mathbf{q}_2$ into our equation for $\mathbf{q}_\alpha$:

$$\begin{aligned}\mathbf{q}_\alpha &= (1-\alpha)\left[(1-\alpha)\mathbf{p}_a + \alpha\mathbf{p}_b\right] + \alpha\left[(1-\alpha)\mathbf{p}_b + \alpha\mathbf{p}_c\right] \\ &= \mathbf{p}_a\left[(1-\alpha)^2\right] + \mathbf{p}_b\left[2\alpha(1-\alpha)\right] + \mathbf{p}_c\left[\alpha^2\right]\end{aligned}$$

Which we can write as the dot product

$$\mathbf{q}_\alpha = \begin{bmatrix} \alpha^2 - 2\alpha + 1 \\ -2\alpha^2 + 2\alpha + 0 \\ \alpha^2 + 0\alpha + 0 \end{bmatrix}^\intercal \begin{bmatrix} \mathbf{p}_a \\ \mathbf{p}_b \\ \mathbf{p}_c \end{bmatrix}$$

The entries of our vector on the left give the basis functions for our spline, which we can plot to show the weight of each control point as a function of progress along our segment:



We can also factor the basis functions for our spline into the product of a spline matrix and different powers of the parameter α:

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_a \\ \mathbf{p}_b \\ \mathbf{p}_c \end{bmatrix}$$

1 Finding the Derivative $\mathbf{s}'(t)$

For now we will consider just a single segment of our quadratic Bézier, so $\alpha = t$. Here we want to find the derivative of our spline with respect to α:

$$\mathbf{s}'(\alpha) = \frac{\partial \mathbf{s}(\alpha)}{\partial \alpha}$$

Our first step will be to write out the equation for our spline as a simple polynomial function of α:

$$\begin{aligned} \mathbf{s}(\alpha) &= \alpha^2 [...] + \alpha [...] + 1 [...] \\ &= \alpha^2 [\mathbf{p}_a - 2\mathbf{p}_b + \mathbf{p}_c] + \alpha [-2\mathbf{p}_a + 2\mathbf{p}_b] + 1 [\mathbf{p}_a] \end{aligned}$$

Now if we take the derivative of this with respect to α we get

$$\mathbf{s}'(\alpha) = 2\alpha [\mathbf{p}_a - 2\mathbf{p}_b + \mathbf{p}_c] + [-2\mathbf{p}_a + 2\mathbf{p}_b]$$

Here we can observe a few things:

- The equation for $\mathbf{s}'(\alpha)$ always yields a vector
- At the start of our spline segment, $\alpha = 0$, the derivative is $2(\mathbf{p}_b - \mathbf{p}_a)$, which points parallel to our first control edge.

- At the end of our spline segment, $\alpha = 1$, the derivative is $2(\mathbf{p}_c - \mathbf{p}_b)$, which points parallel to our second (last) control edge.