

2D Spline Curves

CS 4620 Lecture 18

Motivation: smoothness

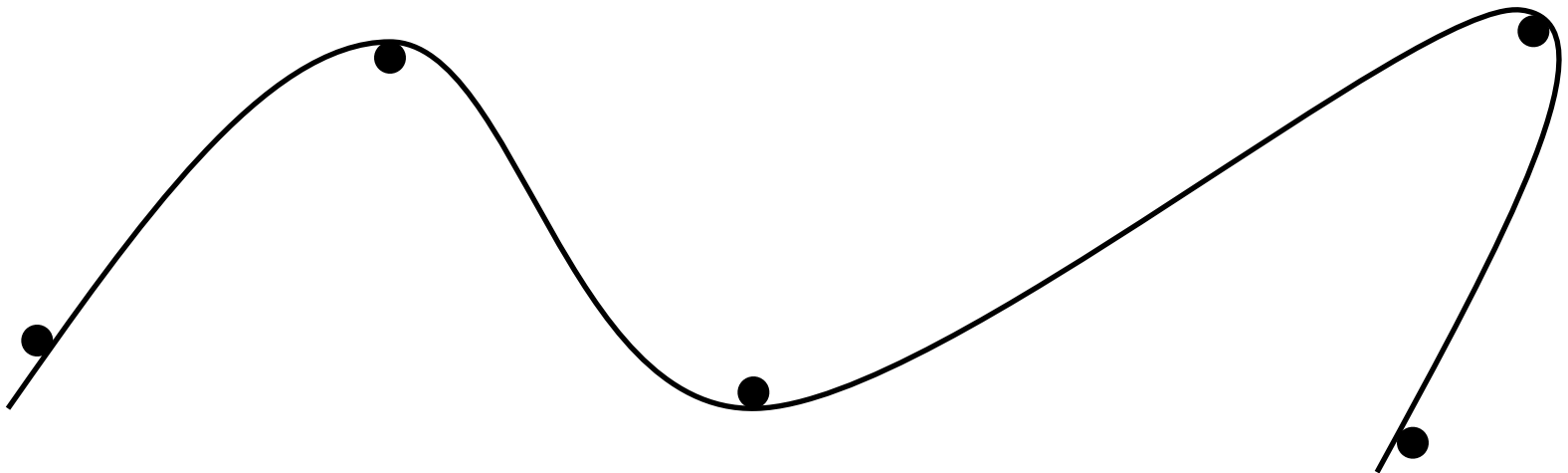
- In many applications we need smooth shapes
 - that is, without discontinuities



- So far we can make
 - things with corners (lines, triangles, squares, rectangles, ...)
 - circles, ellipses, other special shapes (only get you so far!)

Classical approach

- Pencil-and-paper draftsmen also needed smooth curves
- Origin of “spline:” strip of flexible metal
 - held in place by pegs or weights to constrain shape
 - traced to produce smooth contour



Translating into usable math

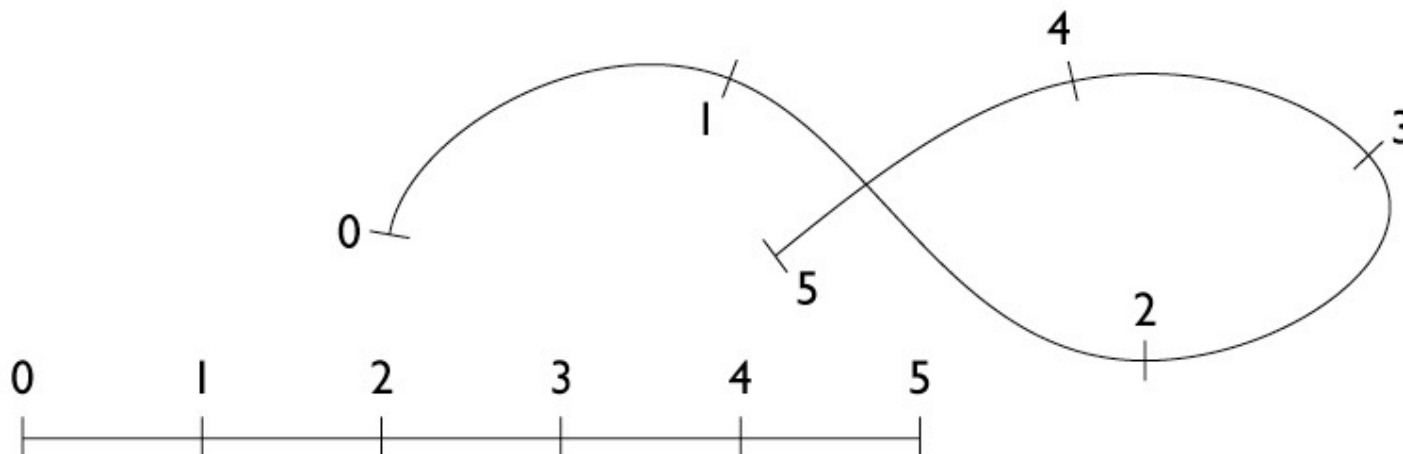
- Smoothness
 - in drafting spline, comes from physical curvature minimization
 - in CG spline, comes from choosing smooth functions
 - usually low-order polynomials
- Control
 - in drafting spline, comes from fixed pegs
 - in CG spline, comes from user-specified *control points*

Defining spline curves

- At the most general they are parametric curves

$$S = \{\mathbf{f}(t) \mid t \in [0, N]\}$$

- For splines, $\mathbf{f}(t)$ is piecewise polynomial
 - for this lecture, the discontinuities are at the integers

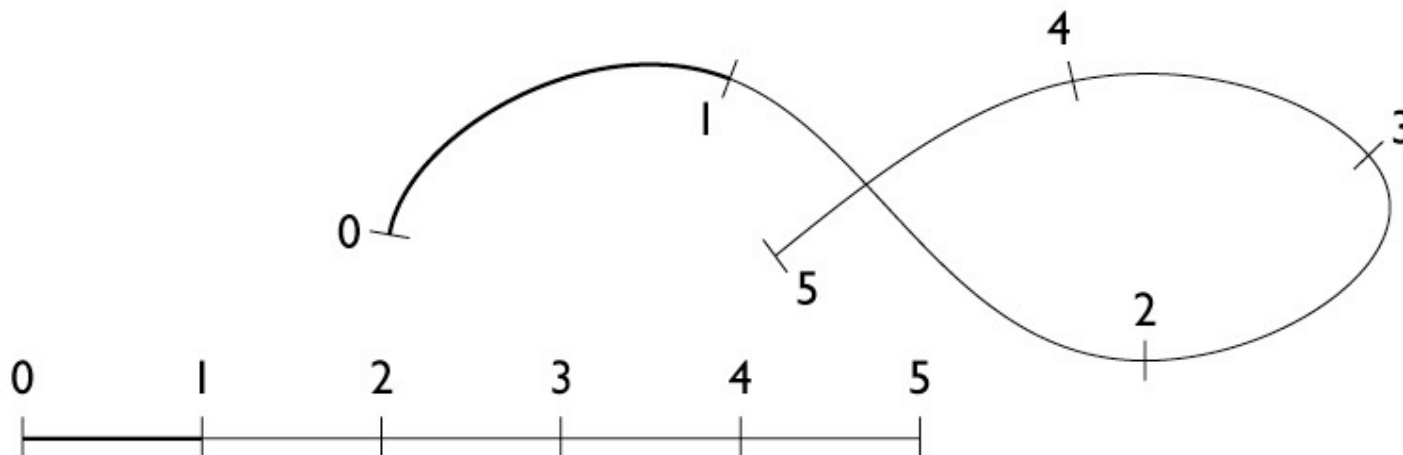


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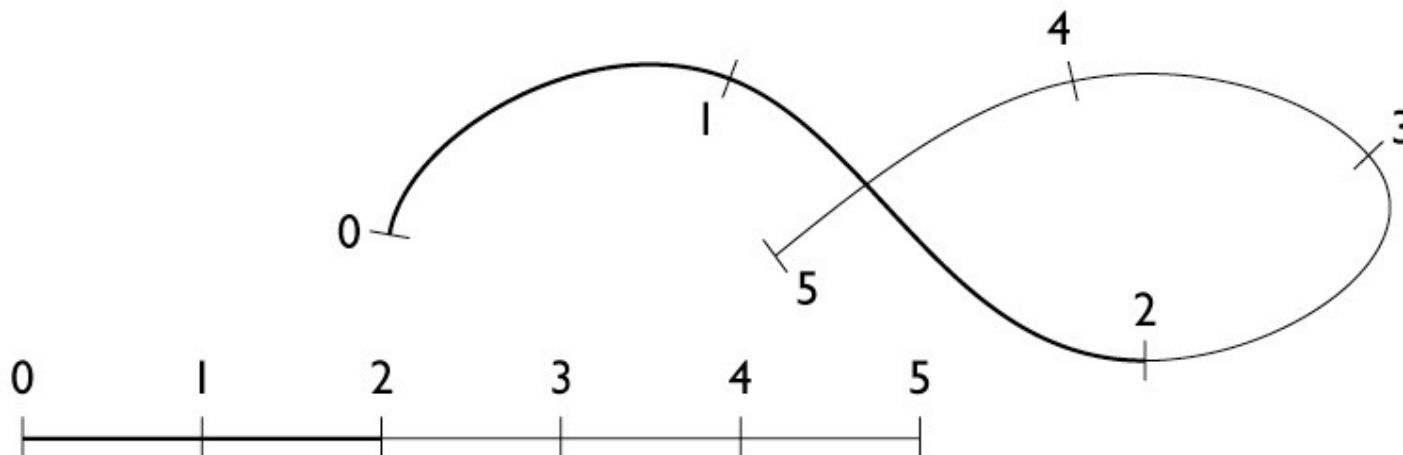


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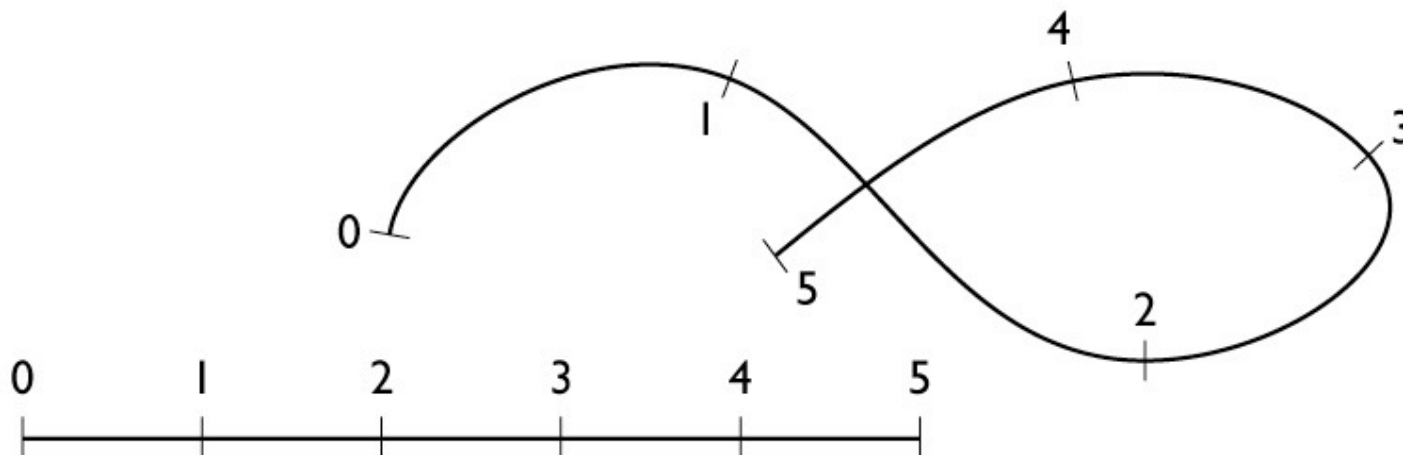


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Defining spline curves

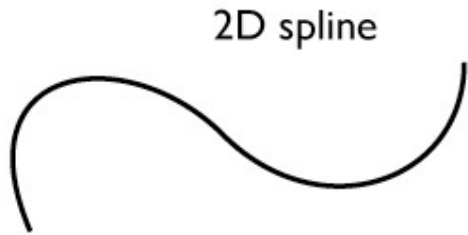
- Generally $\mathbf{f}(t)$ is a piecewise polynomial
 - for this lecture, the discontinuities are at the integers
 - e.g., a cubic spline has the following form over $[k, k + 1]$:

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

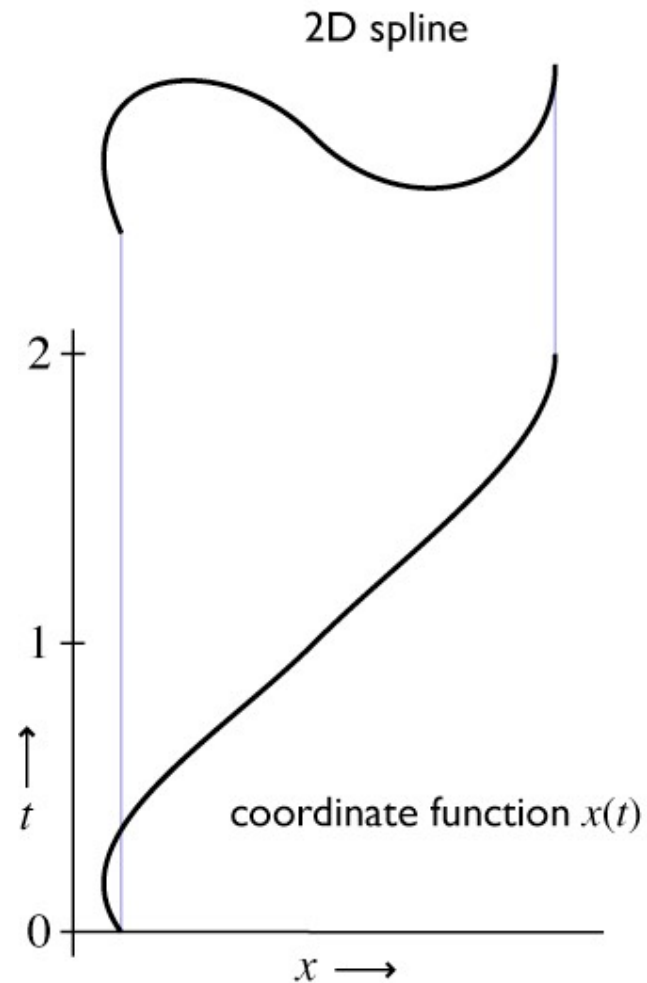
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

- Coefficients are different for every interval

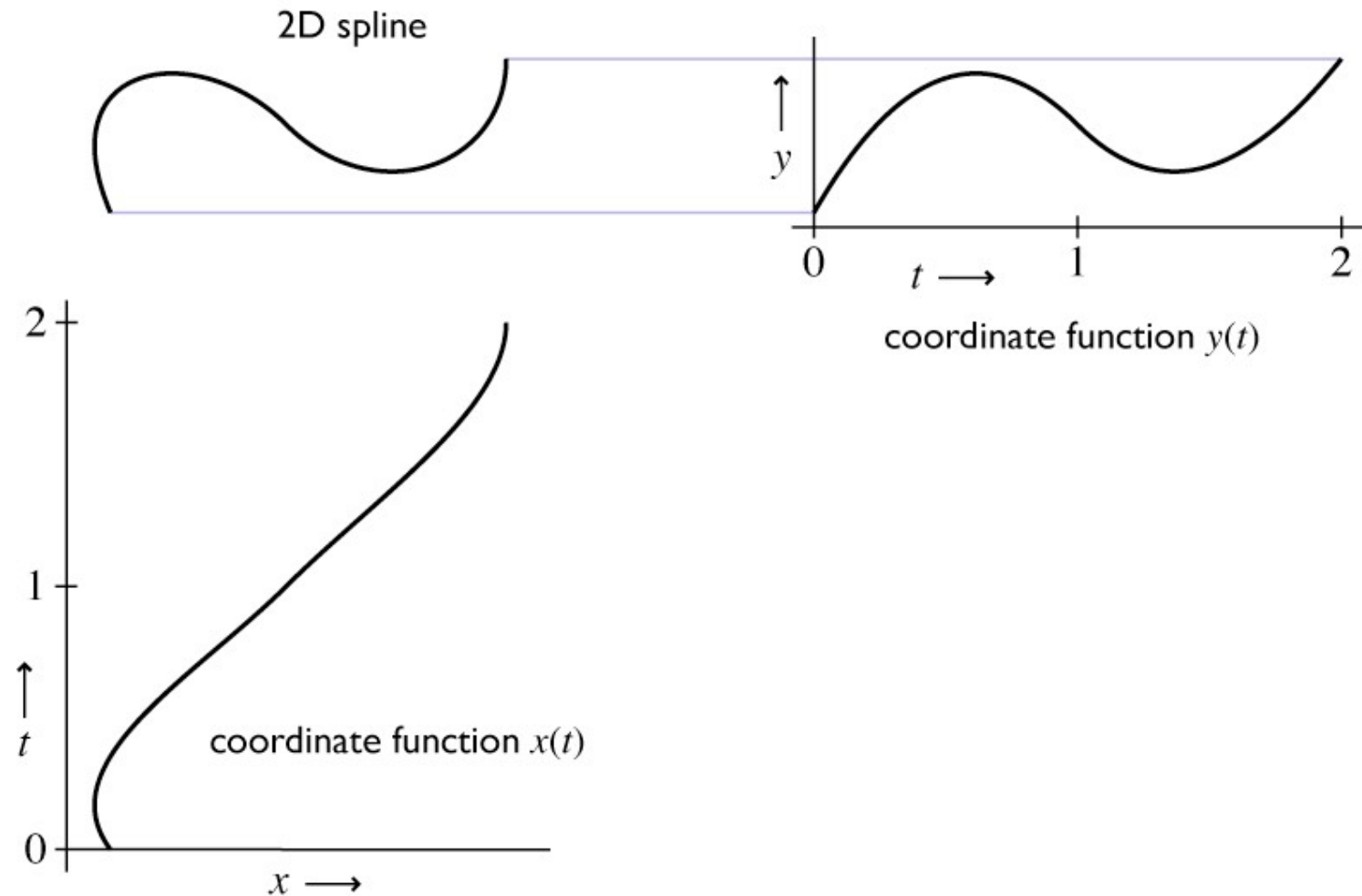
Coordinate functions



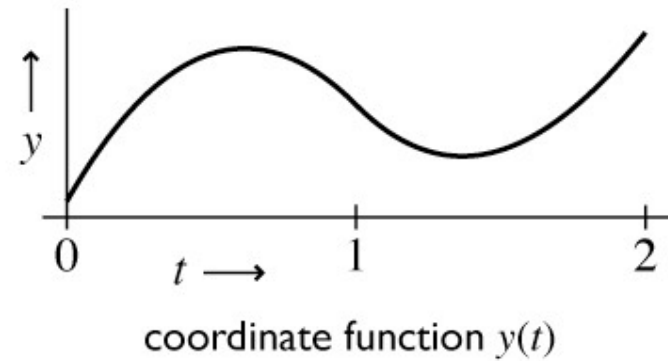
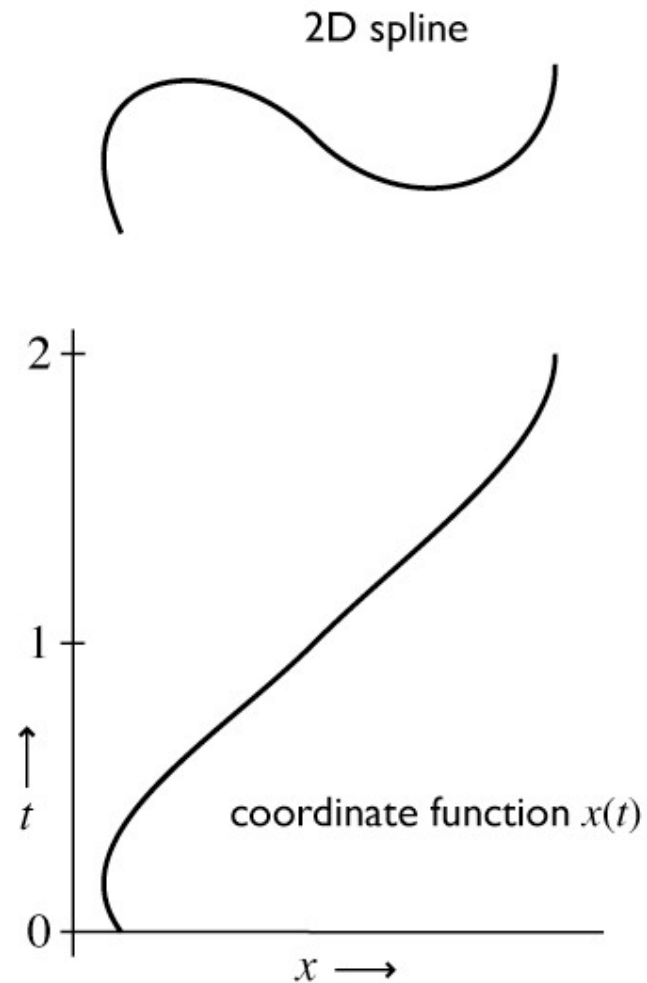
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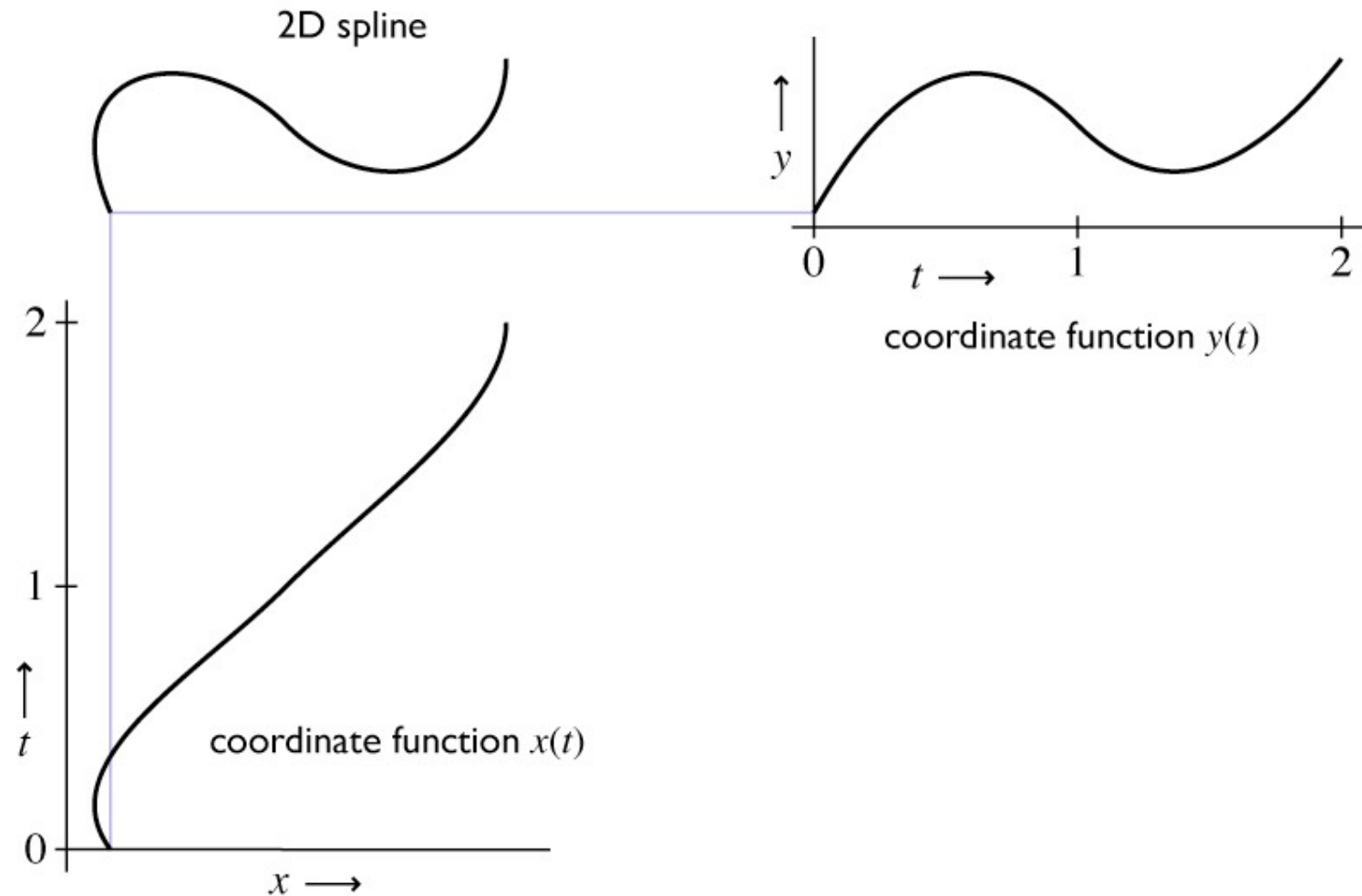
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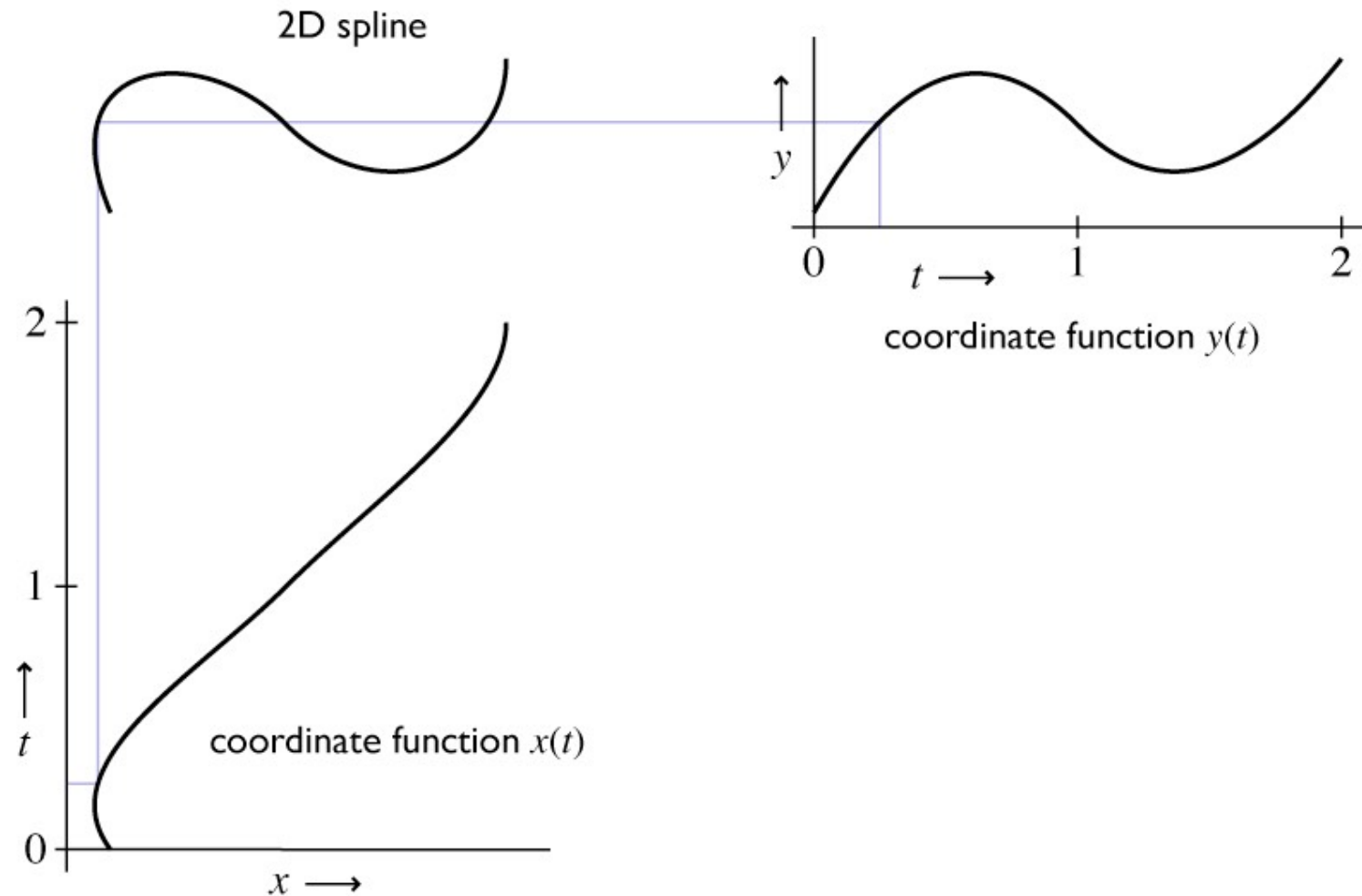
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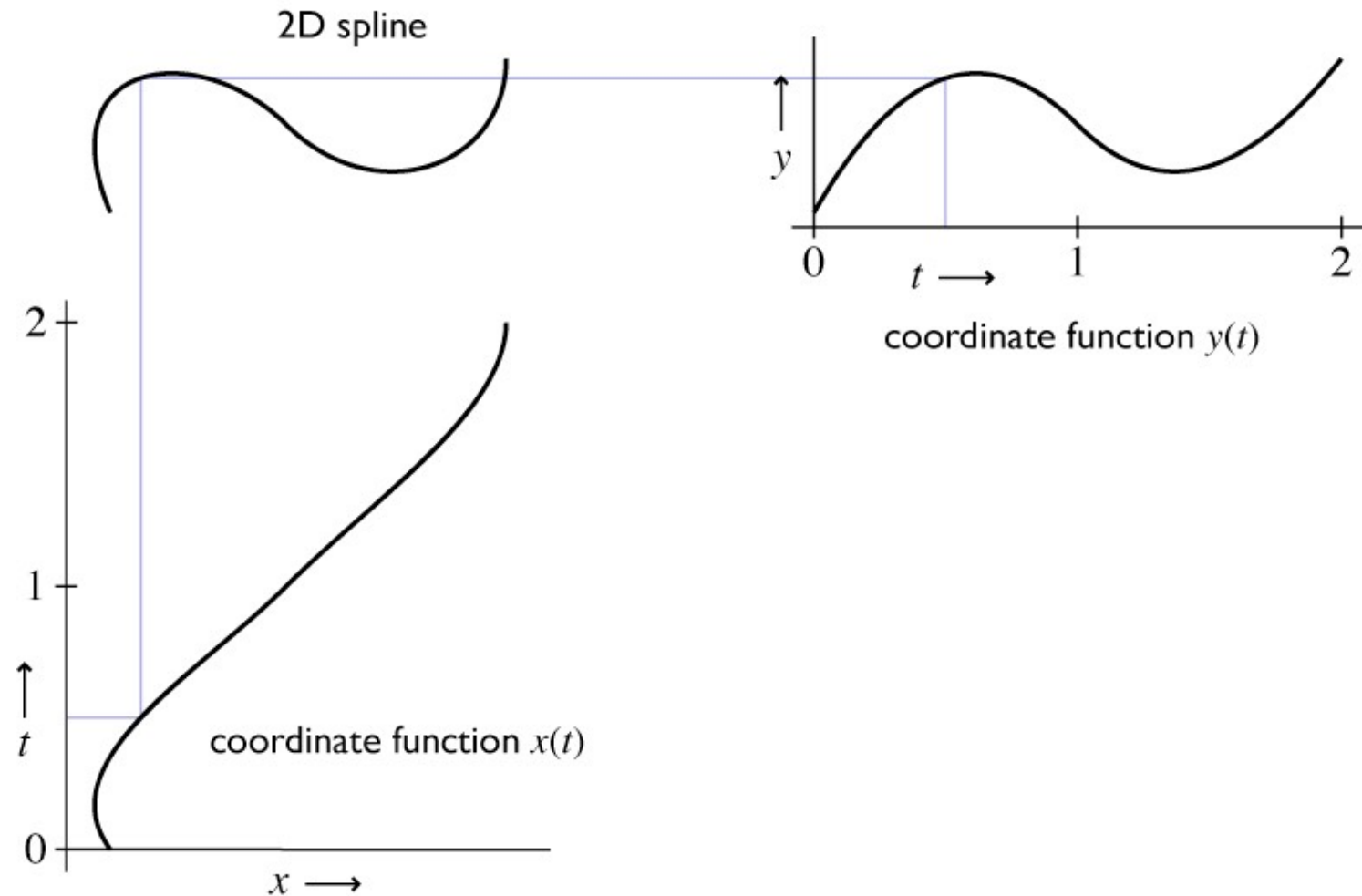
Coordinate functions



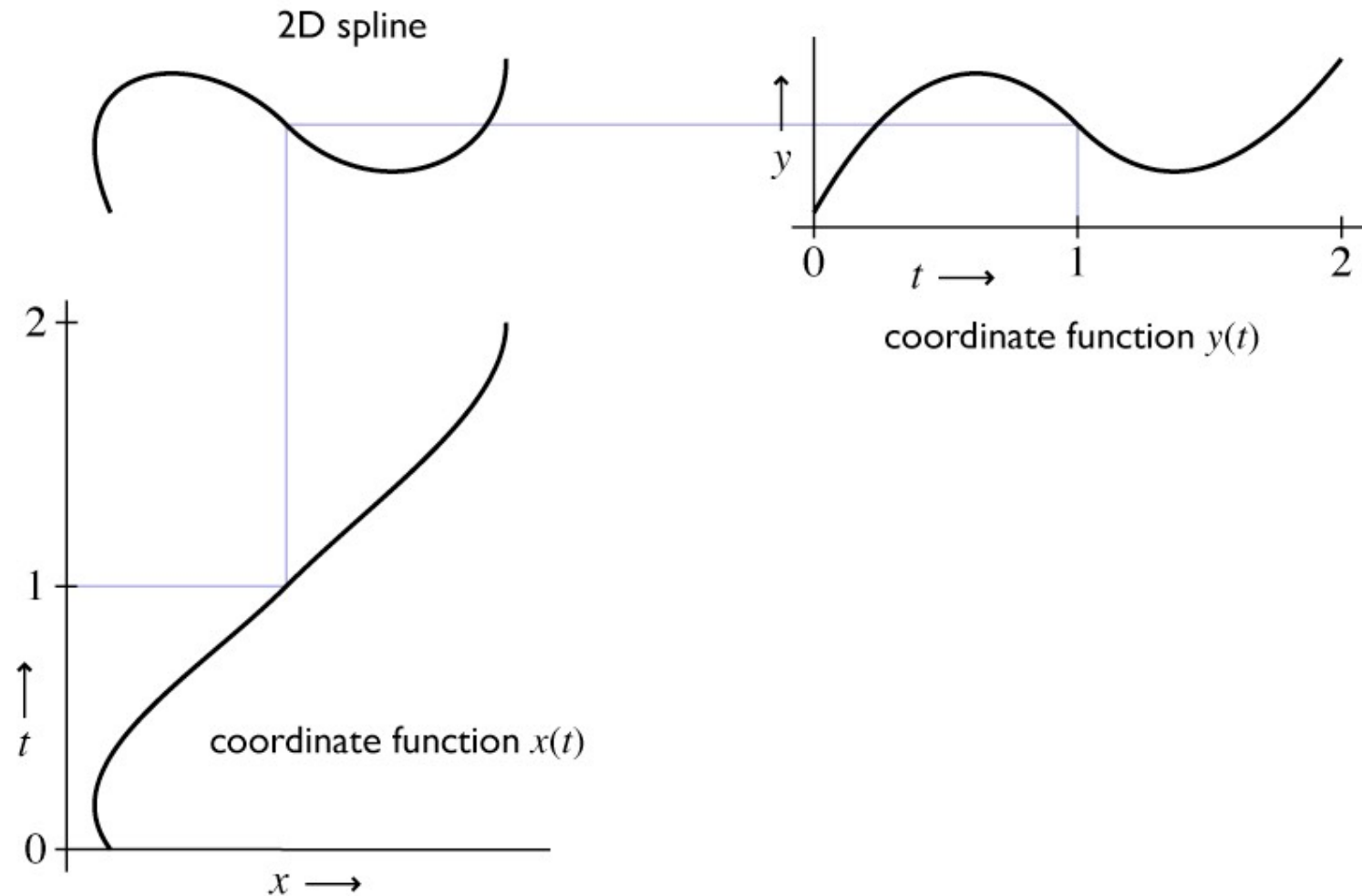
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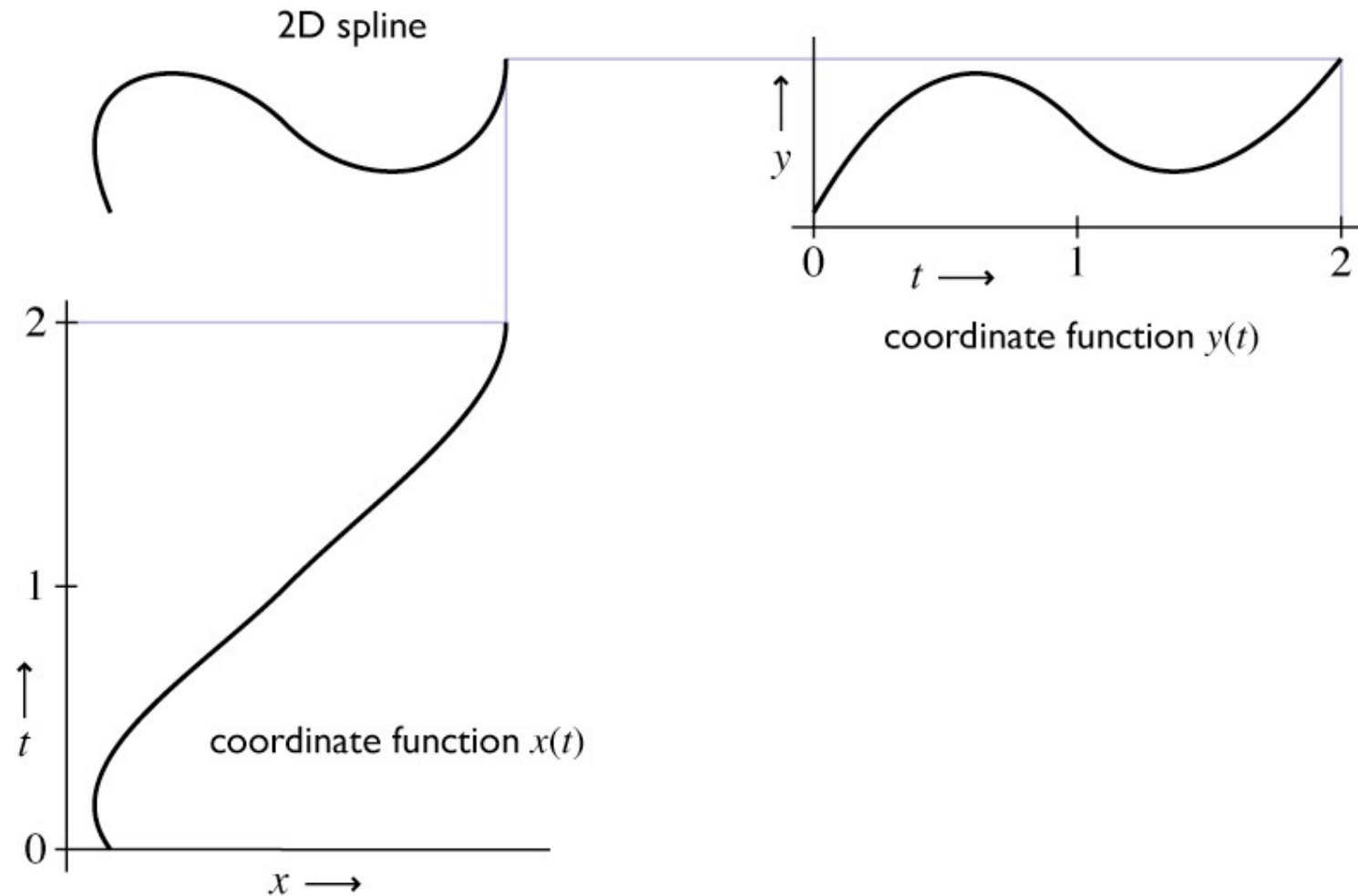
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Coordinate functions

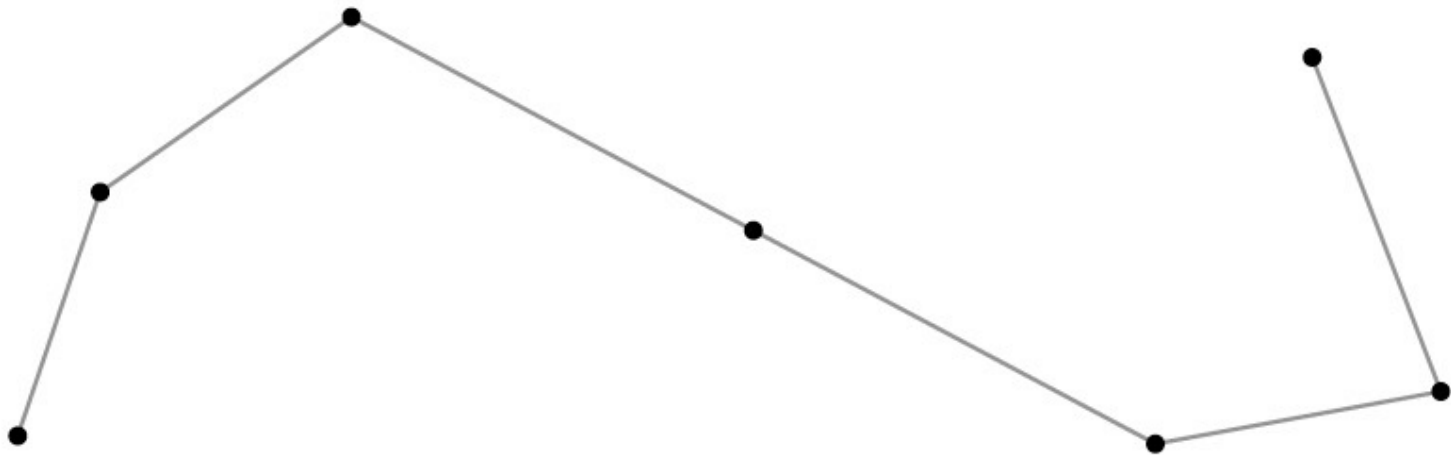


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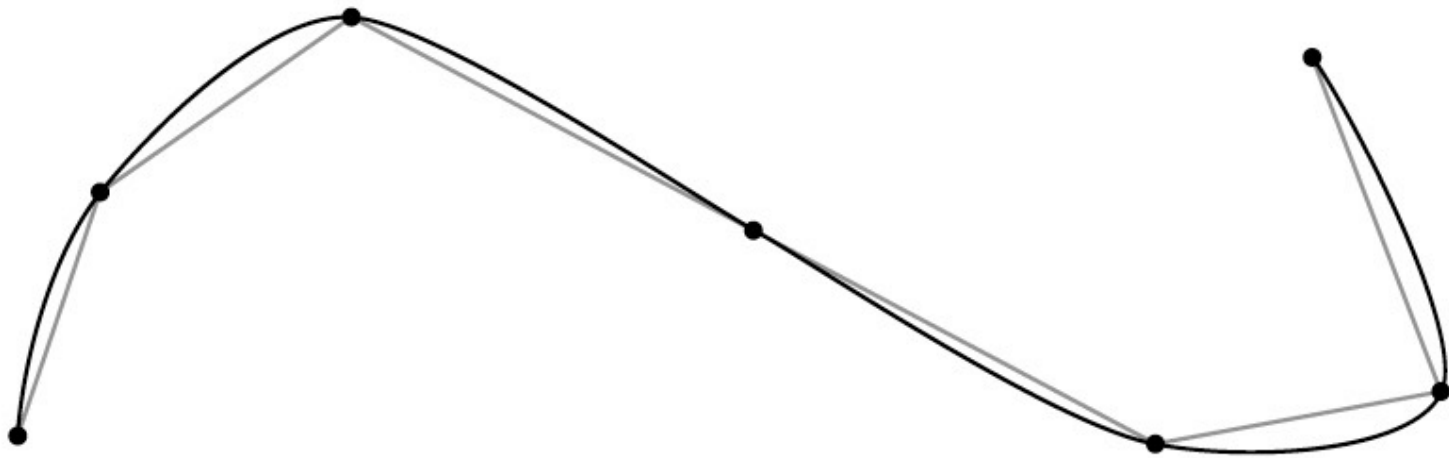
Control of spline curves

- Specified by a sequence of controls (points or vectors)
- Shape is guided by control points (aka control polygon)
 - interpolating: passes through points
 - approximating: merely guided by points
 - some splines interpolate only certain points (e.g. endpoints)



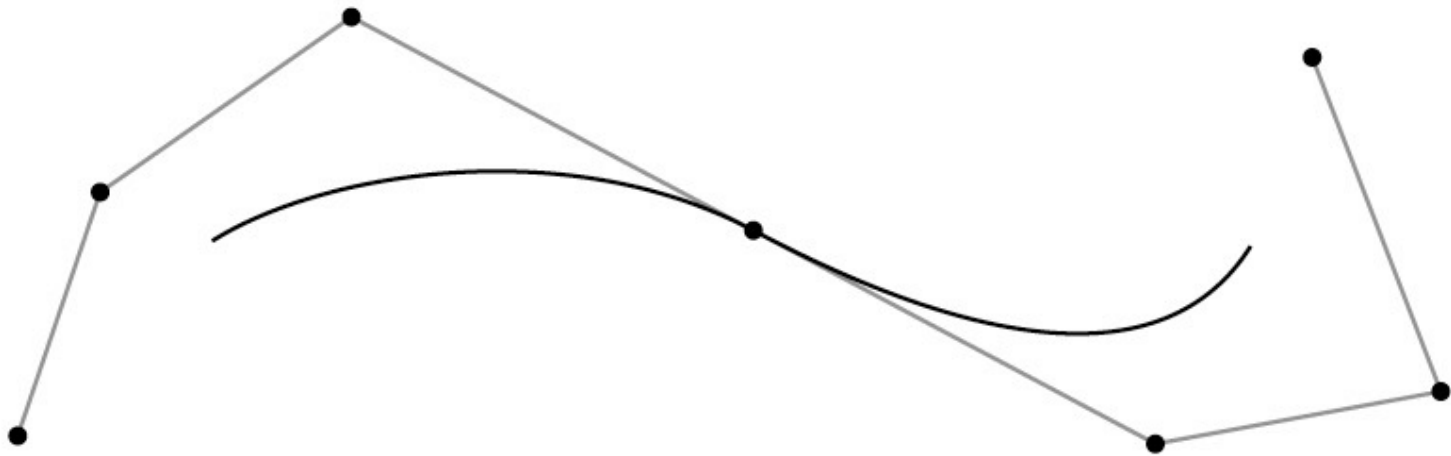
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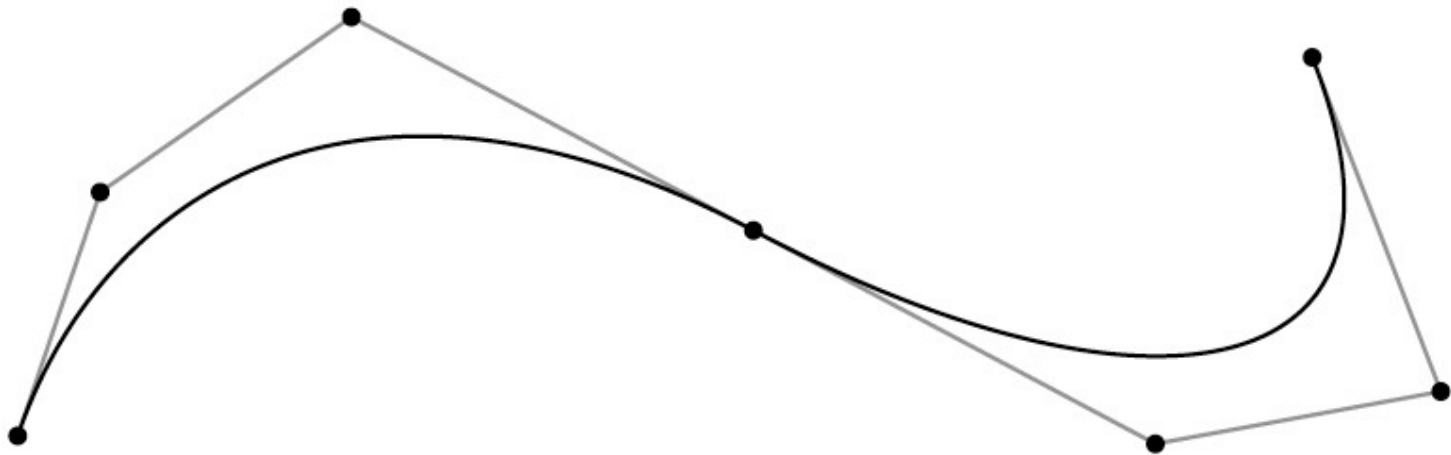
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How splines depend on their controls

- Each coordinate is separate
 - the function $x(t)$ is determined solely by the x coordinates of the control points
 - this means 1D, 2D, 3D, ... curves are all really the same
- Spline curves are **linear** functions of their controls
 - moving a control point two inches to the right moves $x(t)$ twice as far as moving it by one inch
 - $x(t)$, for fixed t , is a linear combination (weighted sum) of the controls' x coordinates
 - $\mathbf{f}(t)$, for fixed t , is a linear combination (weighted sum) of the controls

Plan

1. Spline segments

- how to define a polynomial on $[0,1]$
- ...that has the properties you want
- ...and is easy to control

2. Spline curves

- how to chain together lots of segments
- ...so that the whole curve has the properties you want
- ...and is easy to control

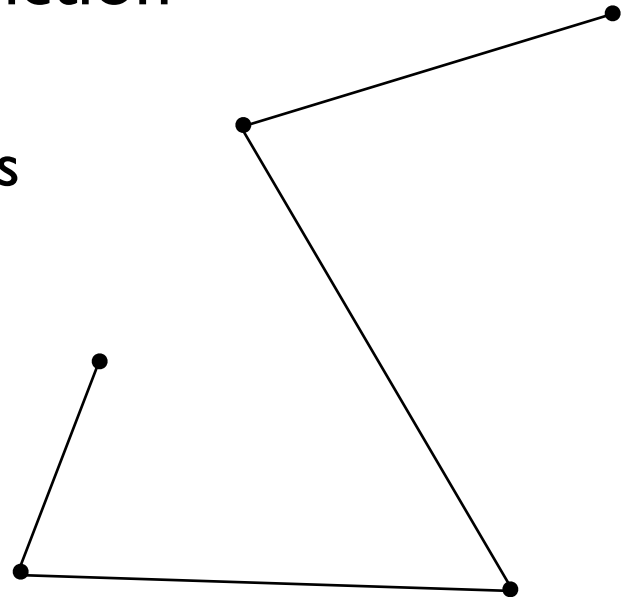
3. Refinement and evaluation

- how to add detail to splines
- how to approximate them with line segments

Spline Segments

Trivial example: piecewise linear

- This spline is just a polygon
 - control points are the vertices
- But we can derive it anyway as an illustration
- Each segment will be a linear function
 - $x(t) = at + b$
 - constraints are values at endpoints
 - $b = x_0$; $a = x_1 - x_0$
 - this is linear interpolation



Trivial example: piecewise linear

- Vector formulation

$$x(t) = (x_1 - x_0)t + x_0$$

$$y(t) = (y_1 - y_0)t + y_0$$

$$\mathbf{f}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0$$

- Matrix formulation

$$\mathbf{f}(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

Trivial example: piecewise linear

- Basis function formulation
 - regroup expression by **p** rather than *t*

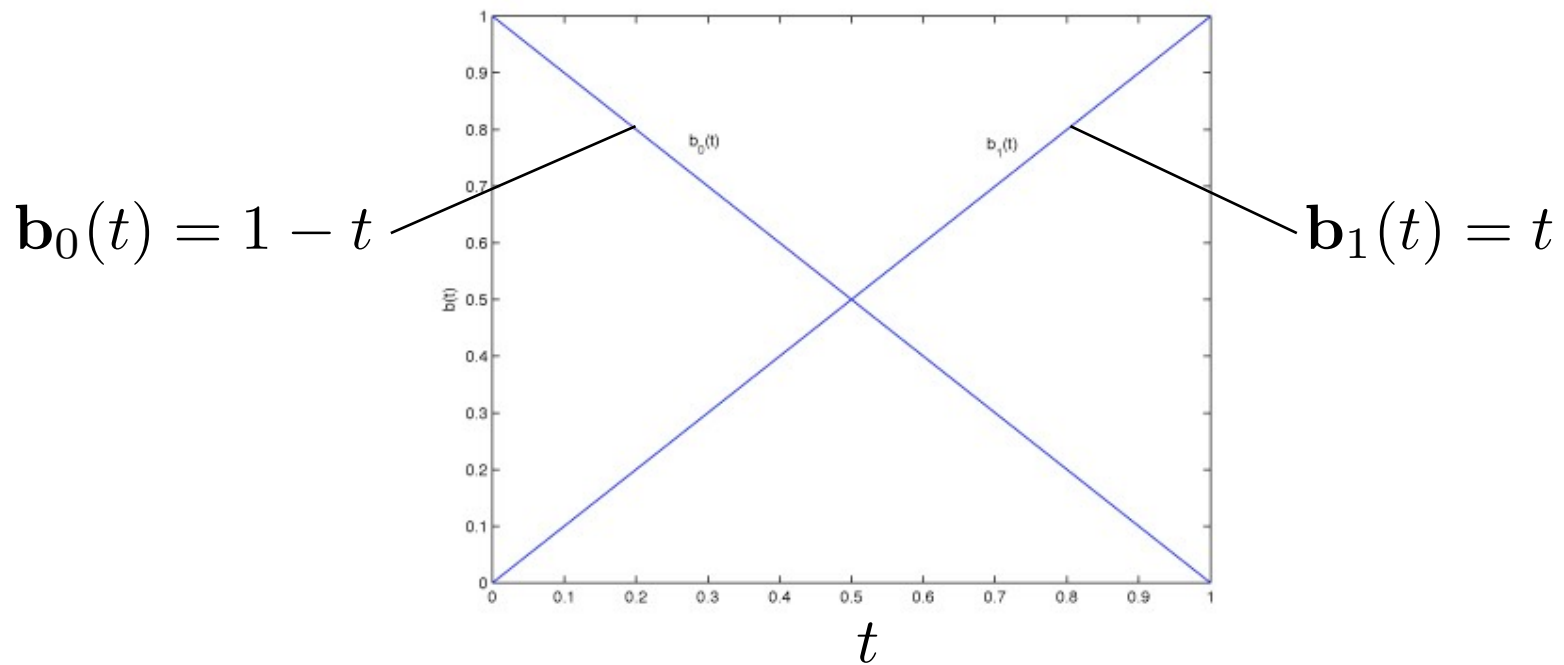
$$\begin{aligned}\mathbf{f}(t) &= (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0 \\ &= (1 - t)\mathbf{p}_0 + t\mathbf{p}_1\end{aligned}$$

- interpretation in matrix viewpoint

$$\mathbf{f}(t) = \left(\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

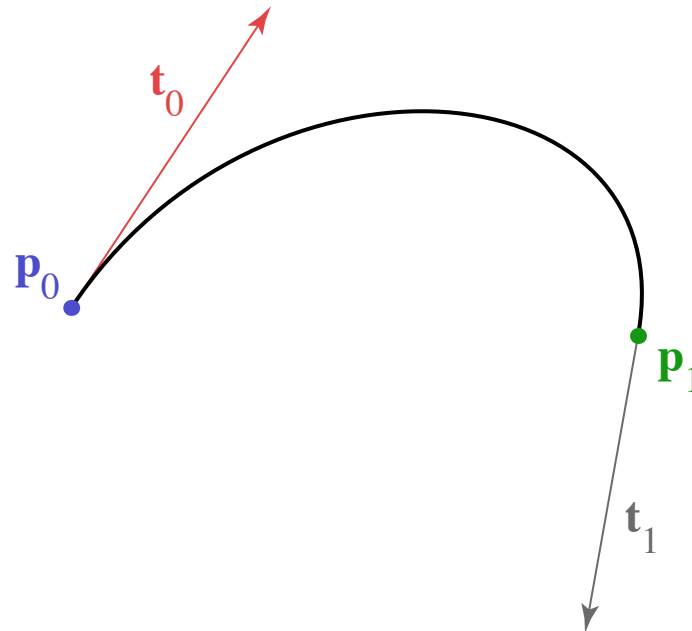
Trivial example: piecewise linear

- Vector blending formulation: “average of points”
 - blending functions: contribution of each point as t changes



Hermite splines

- Less trivial example
- Form of curve: piecewise cubic
- Constraints: endpoints and tangents (derivatives)



Hermite splines

- Solve constraints to find coefficients

$$x(t) = at^3 + bt^2 + ct + d$$

$$x'(t) = 3at^2 + 2bt + c$$

$$x(0) = x_0 = d$$

$$x(1) = x_1 = a + b + c + d$$

$$x'(0) = x'_0 = c$$

$$x'(1) = x'_1 = 3a + 2b + c$$

$$d = x_0$$

$$c = x'_0$$

$$a = 2x_0 - 2x_1 + x'_0 + x'_1$$

$$b = -3x_0 + 3x_1 - 2x'_0 - x'_1$$

Matrix form of spline

$$\mathbf{f}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

$$\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$\mathbf{f}(t) = b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1 + b_2(t)\mathbf{p}_2 + b_3(t)\mathbf{p}_3$$

Matrix form of spline

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Hermite splines

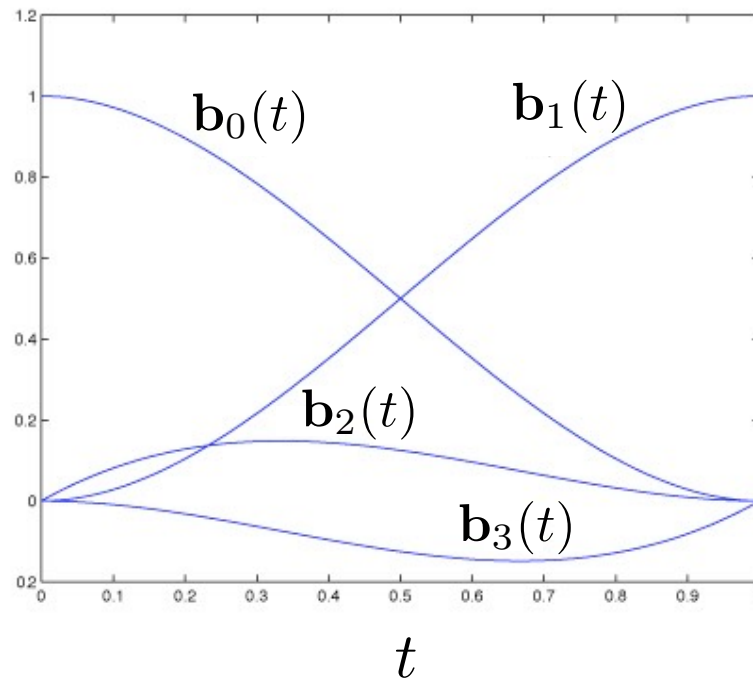
- Matrix form is much simpler

$$\mathbf{f}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{t}_0 \\ \mathbf{t}_1 \end{bmatrix}$$

- coefficients = rows
- basis functions = columns
 - note the two **p** columns sum to $[0 \ 0 \ 0 \ 1]^T$

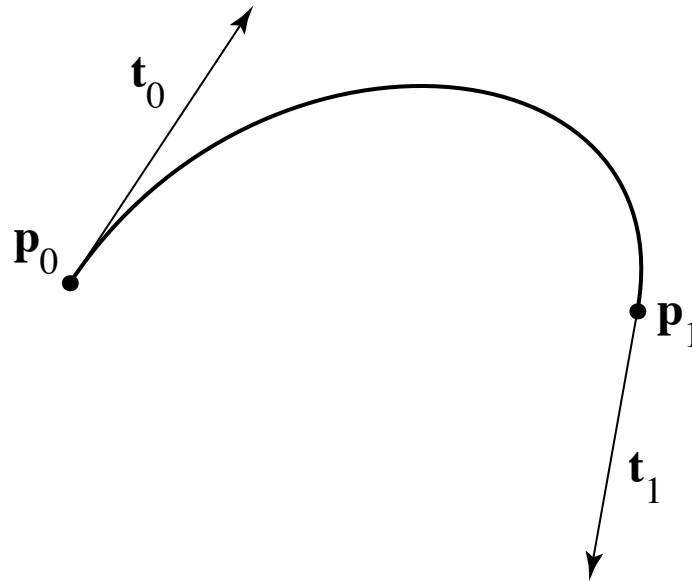
Hermite splines

- Hermite blending functions



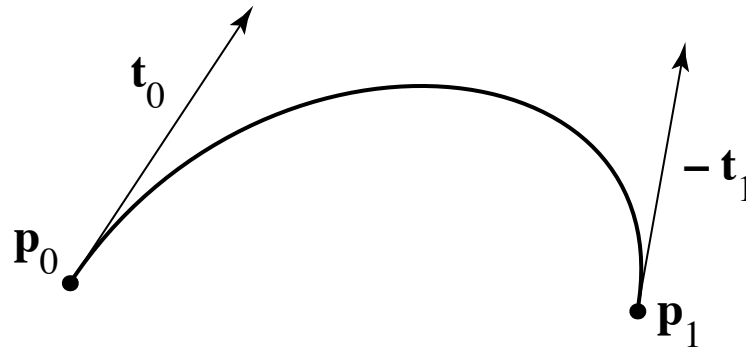
Hermite to Bézier

- Mixture of points and vectors is awkward
- Specify tangents as differences of points



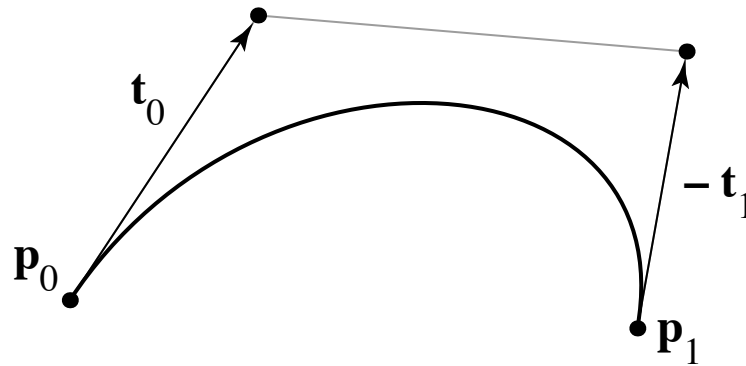
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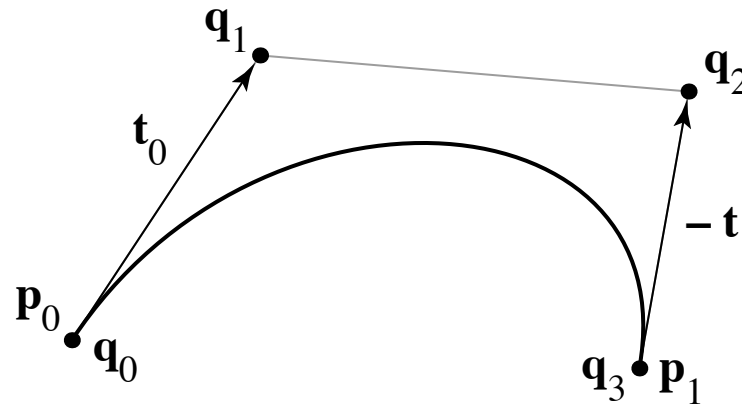
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Hermite to Bézier

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I'm calling these points **q** just for this slide and the next one.

- note derivative is defined as 3 times offset
 - reason is illustrated by linear case

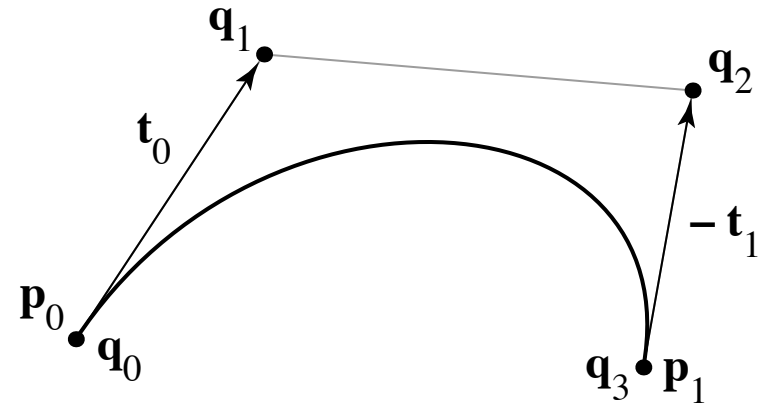
Hermite to Bézier

$$\mathbf{p}_0 = \mathbf{q}_0$$

$$\mathbf{p}_1 = \mathbf{q}_3$$

$$\mathbf{t}_0 = 3(\mathbf{q}_1 - \mathbf{q}_0)$$

$$\mathbf{t}_1 = 3(\mathbf{q}_3 - \mathbf{q}_2)$$



$$\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_0 \\ \mathbf{v}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{bmatrix}$$

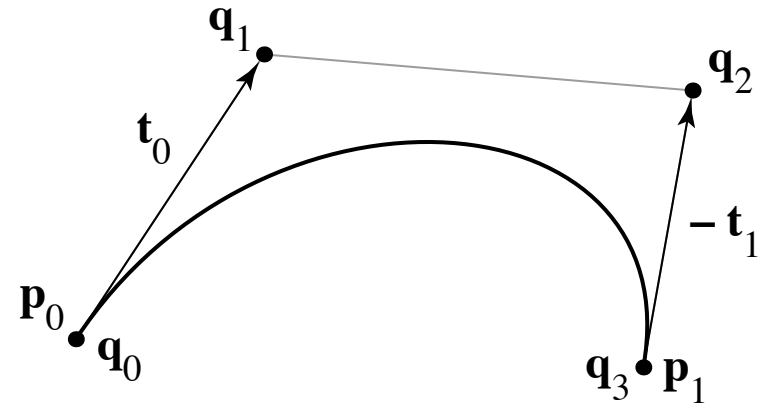
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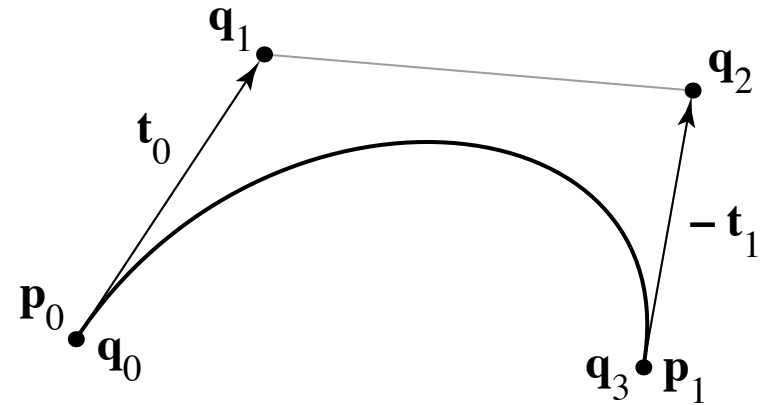
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$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{bmatrix}$$

Bézier matrix

$$\mathbf{f}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

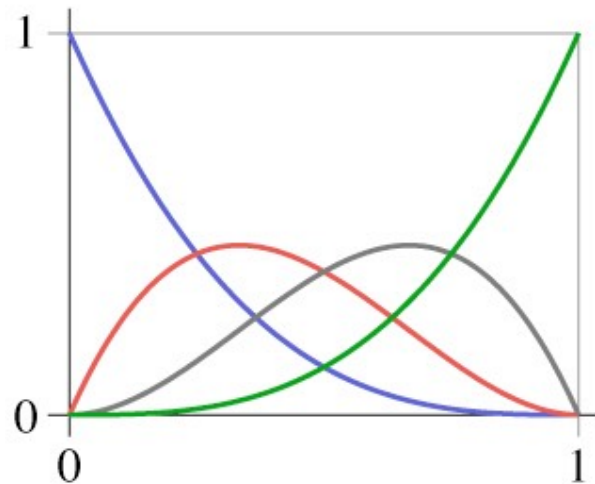
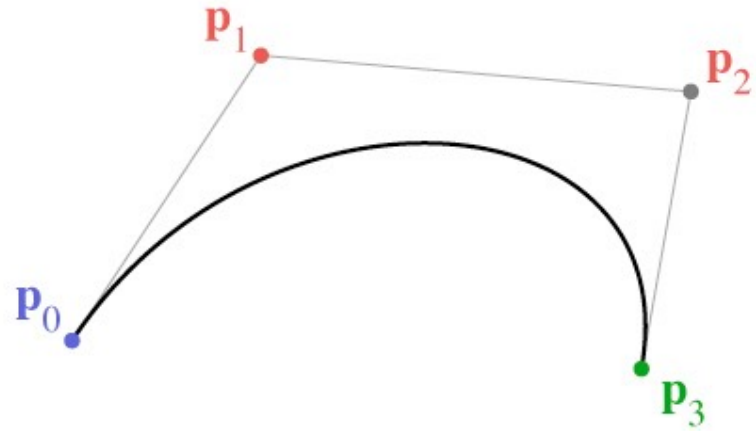
- note that the basis functions are the Bernstein polynomials

$$b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

and that defines Bézier curves for any degree

The $\mathbf{p}_k$ column of the matrix defines the polynomial $b_{3,k}(t)$

Bézier basis



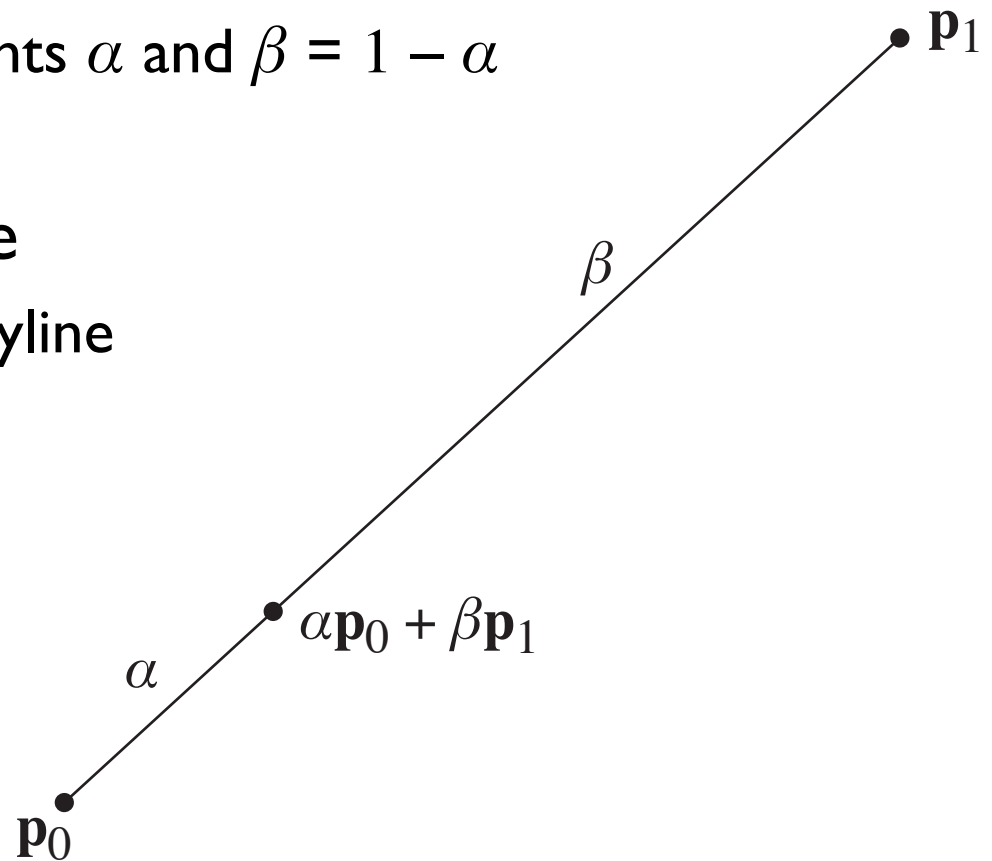
Another way to Bézier segments

- A really boring spline segment: $f(t) = p_0$
 - it only has one control point
 - the curve stays at that point for the whole time
- Only good for building a *piecewise constant* spline
 - a.k.a. a set of points

• p_0

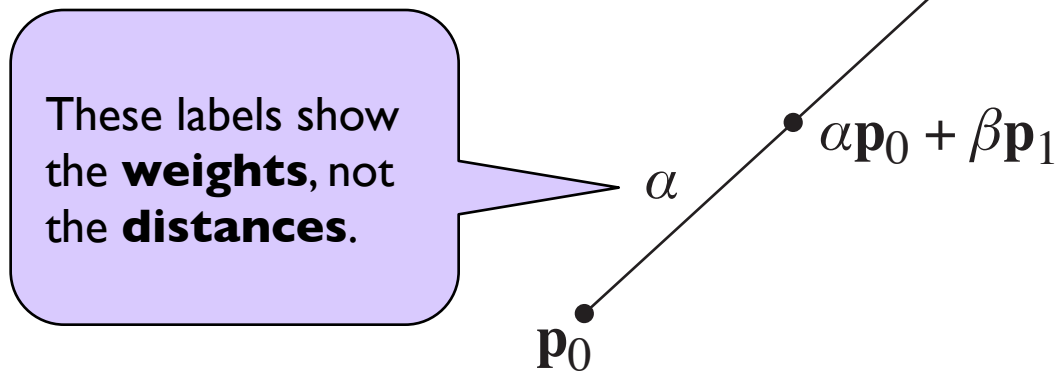
Another way to Bézier segments

- A piecewise linear spline segment
 - two control points per segment
 - blend them with weights α and $\beta = 1 - \alpha$
- Good for building a piecewise linear spline
 - a.k.a. a polygon or polyline



Another way to Bézier segments

- A piecewise linear spline segment
 - two control points per segment
 - blend them with weights α and $\beta = 1 - \alpha$
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Another way to Bézier segments

- A linear blend of two piecewise linear segments
 - three control points now
 - interpolate on both segments using α and β
 - blend the results with the same weights
- makes a quadratic spline segment
 - finally, a curve!

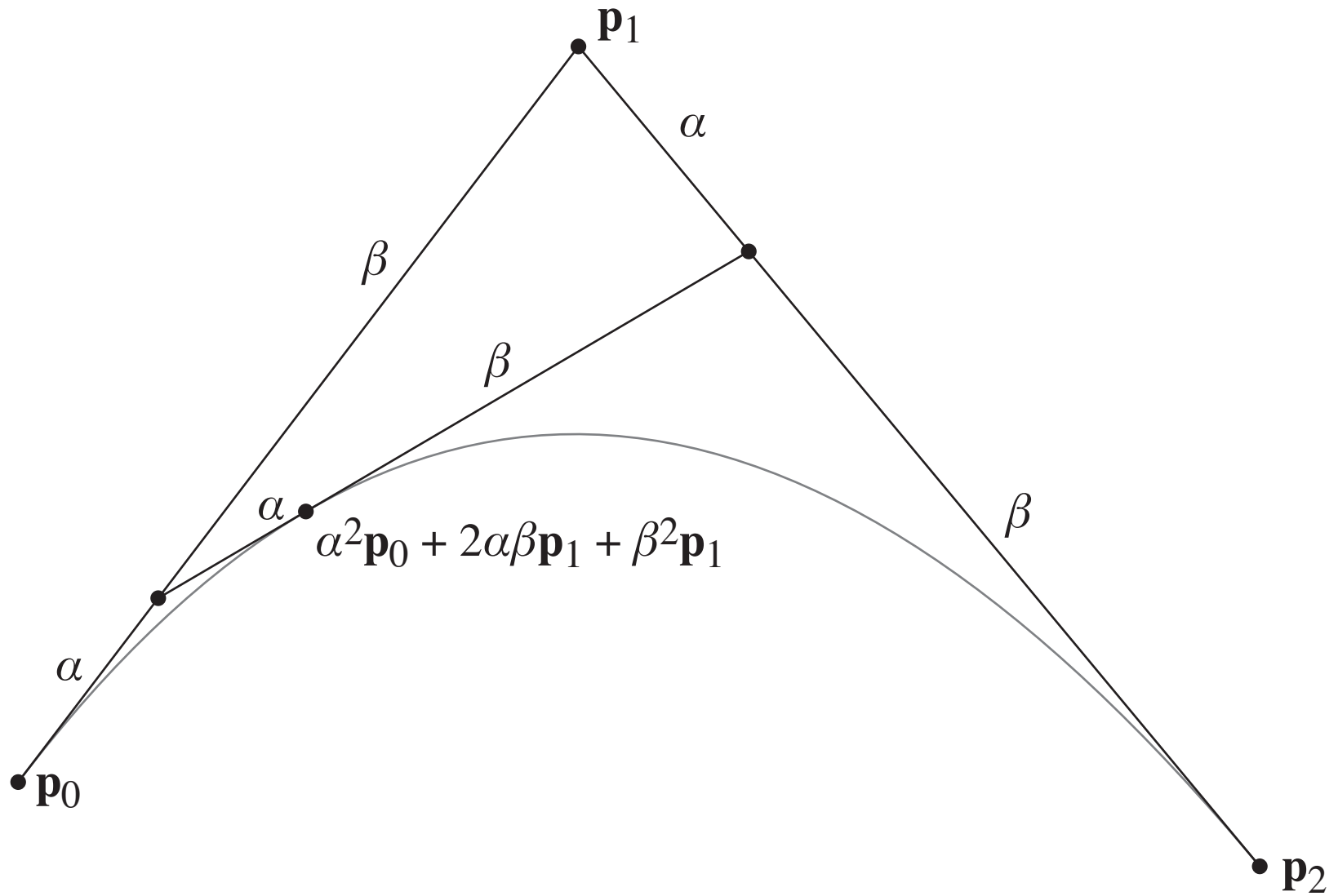
$$\mathbf{p}_{1,0} = \alpha \mathbf{p}_0 + \beta \mathbf{p}_1$$

$$\mathbf{p}_{1,1} = \alpha \mathbf{p}_1 + \beta \mathbf{p}_2$$

$$\mathbf{p}_{2,0} = \alpha \mathbf{p}_{1,0} + \beta \mathbf{p}_{1,1}$$

$$= \alpha \alpha \mathbf{p}_0 + \alpha \beta \mathbf{p}_1 + \beta \alpha \mathbf{p}_1 + \beta \beta \mathbf{p}_2$$

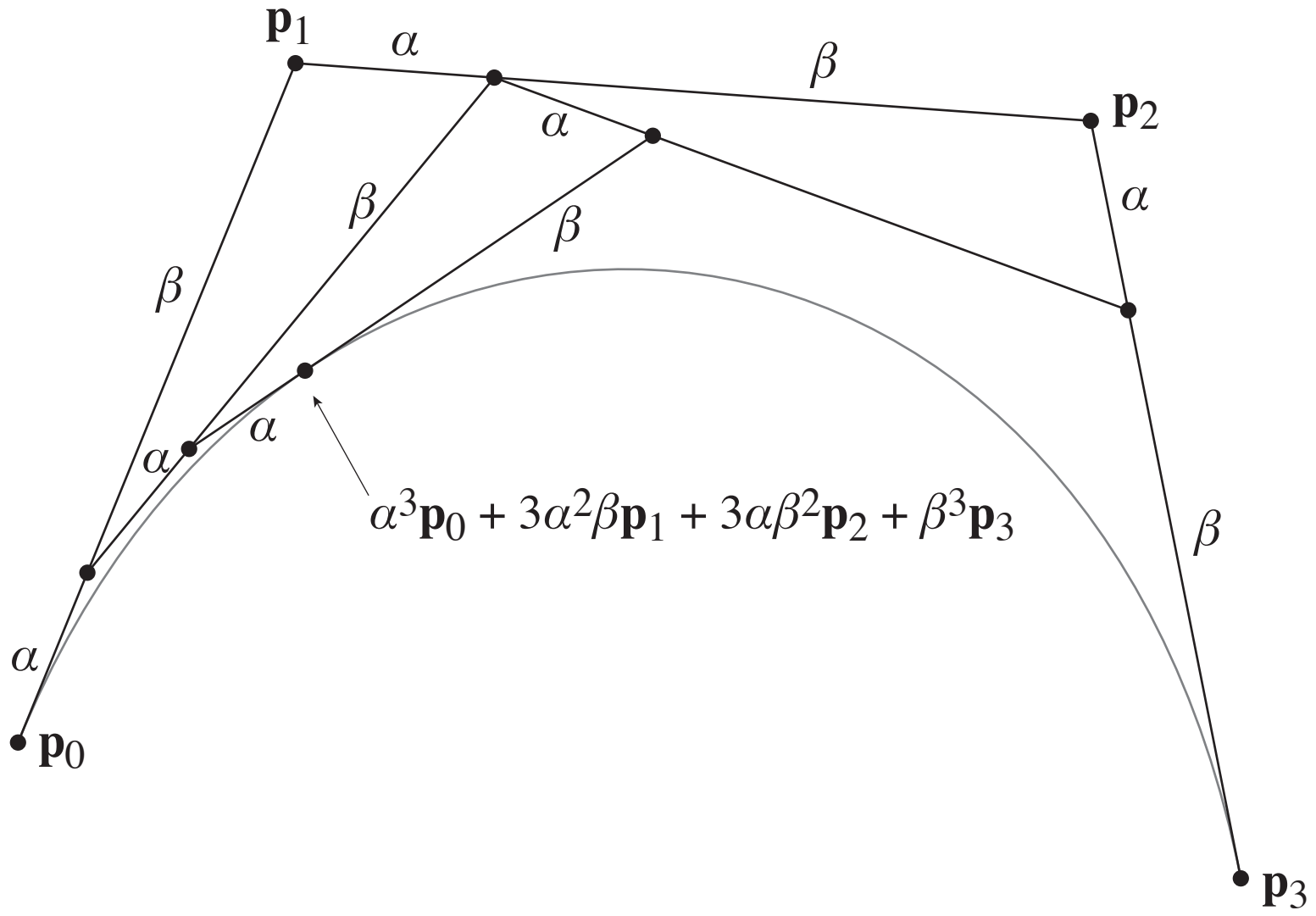
$$= \alpha^2 \mathbf{p}_0 + 2\alpha\beta \mathbf{p}_1 + \beta^2 \mathbf{p}_2$$



Another way to Bézier segments

- Cubic segment: blend of two quadratic segments
 - four control points now (overlapping sets of 3)
 - interpolate on each quadratic using α and β
 - blend the results with the same weights
- makes a cubic spline segment
 - this is the familiar one for graphics—but you can keep going

$$\begin{aligned}\mathbf{p}_{3,0} &= \alpha \mathbf{p}_{2,0} + \beta \mathbf{p}_{2,1} \\ &= \alpha \alpha \alpha \mathbf{p}_0 + \alpha \alpha \beta \mathbf{p}_1 + \alpha \beta \alpha \mathbf{p}_1 + \alpha \beta \beta \mathbf{p}_2 \\ &\quad \beta \alpha \alpha \mathbf{p}_1 + \beta \alpha \beta \mathbf{p}_2 + \beta \beta \alpha \mathbf{p}_2 + \beta \beta \beta \mathbf{p}_3 \\ &= \alpha^3 \mathbf{p}_0 + 3\alpha^2 \beta \mathbf{p}_1 + 3\alpha \beta^2 \mathbf{p}_2 + \beta^3 \mathbf{p}_3\end{aligned}$$



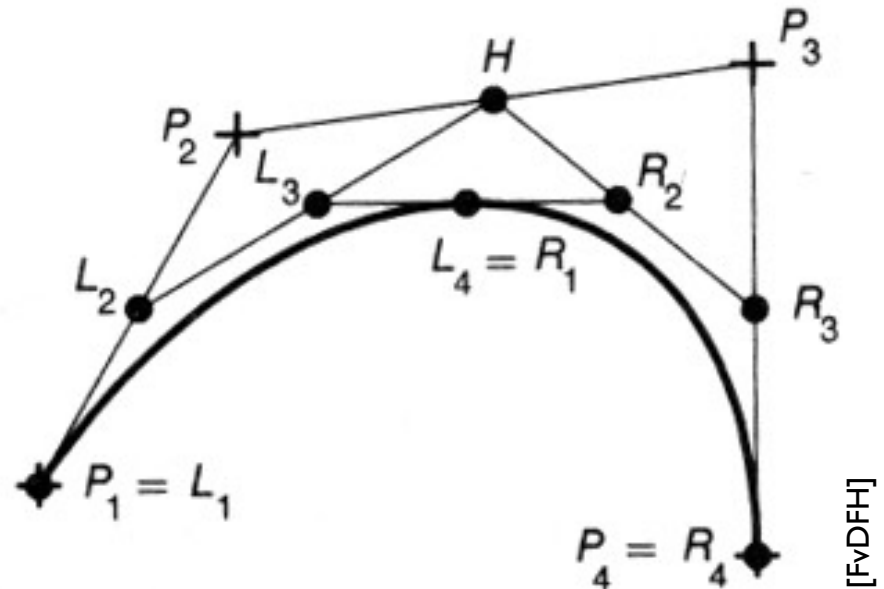
de Casteljau's algorithm

- A recurrence for computing points on Bézier spline segments:

$$\mathbf{p}_{0,i} = \mathbf{p}_i$$

$$\mathbf{p}_{n,i} = \alpha \mathbf{p}_{n-1,i} + \beta \mathbf{p}_{n-1,i+1}$$

- Cool additional feature:
also subdivides
the segment into two
shorter ones

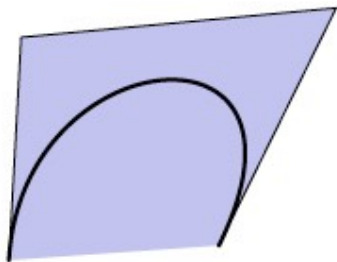


Cubic Bézier splines

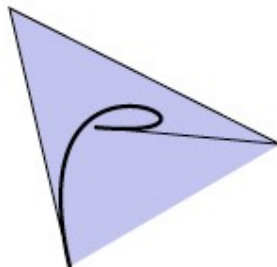
- Very widely used type, especially in 2D
 - e.g. it is a primitive in PostScript/PDF
- Can represent smooth curves with corners
- Nice de Casteljau recurrence for evaluation
- Can easily add points at any position
- Illustrator demo

Spline segment properties

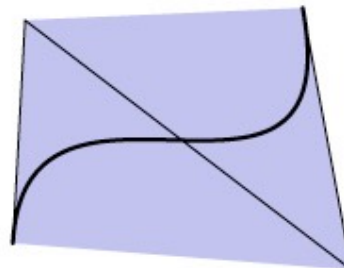
- Convex hull property
 - convex hull = smallest convex region containing points
 - think of a rubber band around some pins
 - some splines stay inside convex hull of control points
 - make clipping, culling, picking, etc. simpler



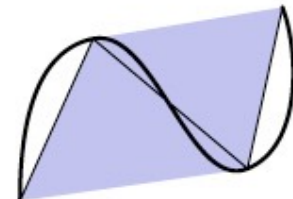
YES



YES



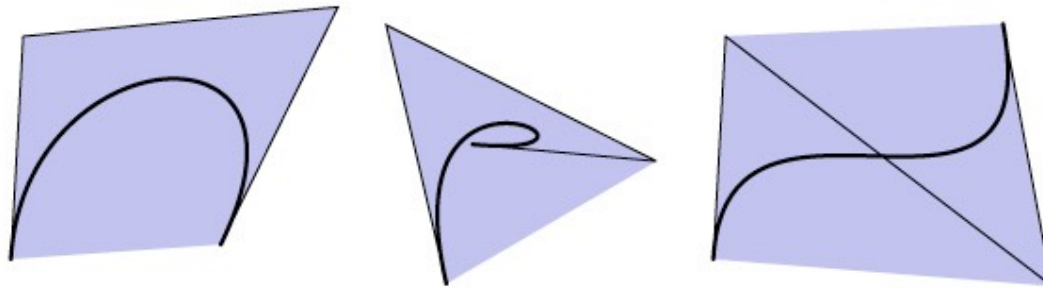
YES



NO

Convex hull

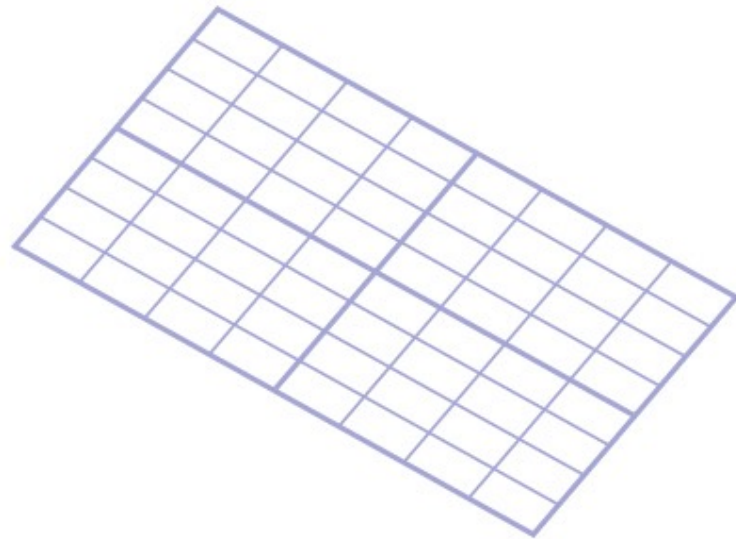
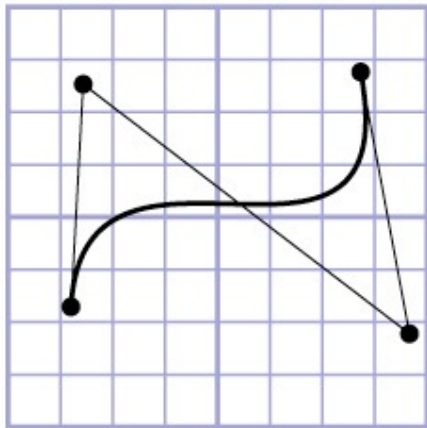
- If basis functions are all positive, the spline has the convex hull property
 - we're still requiring them to sum to 1



- if any basis function is ever negative, no convex hull prop.
 - proof: take the other three points at the same place

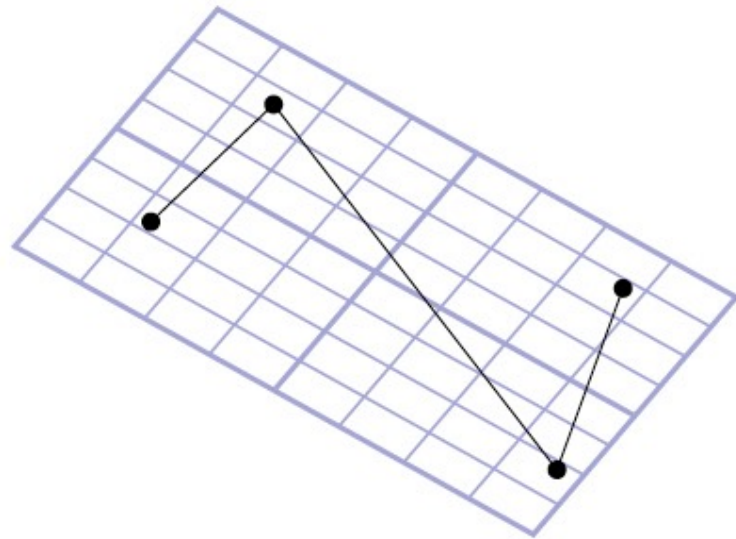
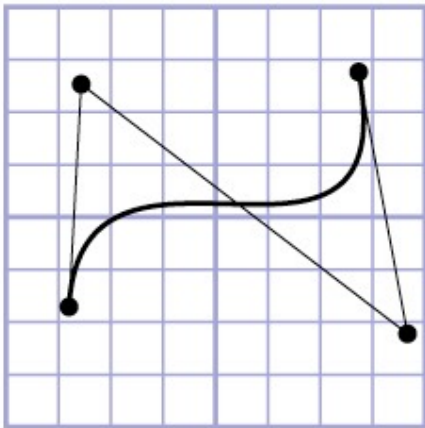
Affine invariance

- Transforming the control points is the same as transforming the curve
 - true for all commonly used splines
 - extremely convenient in practice...



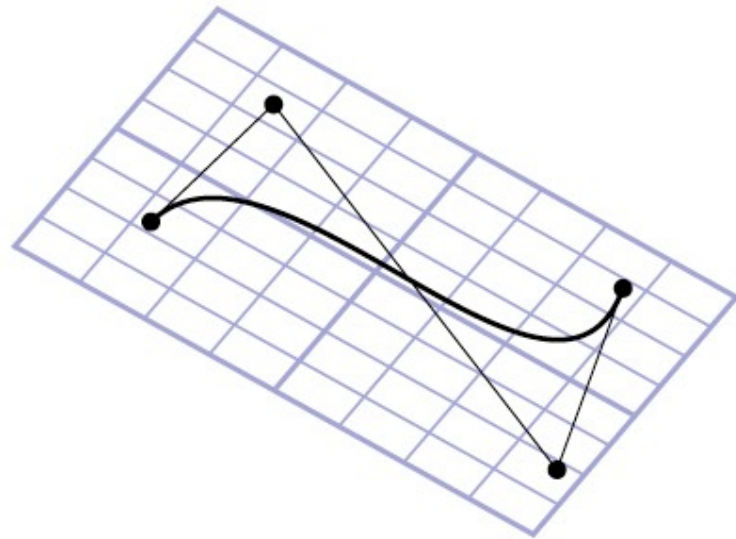
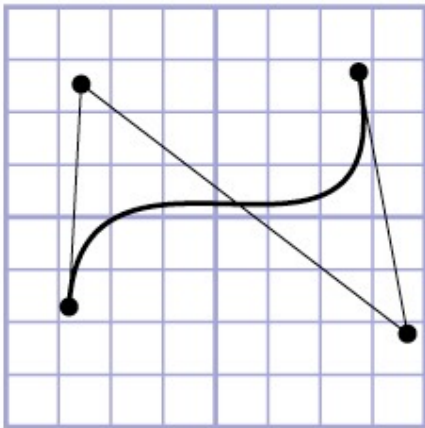
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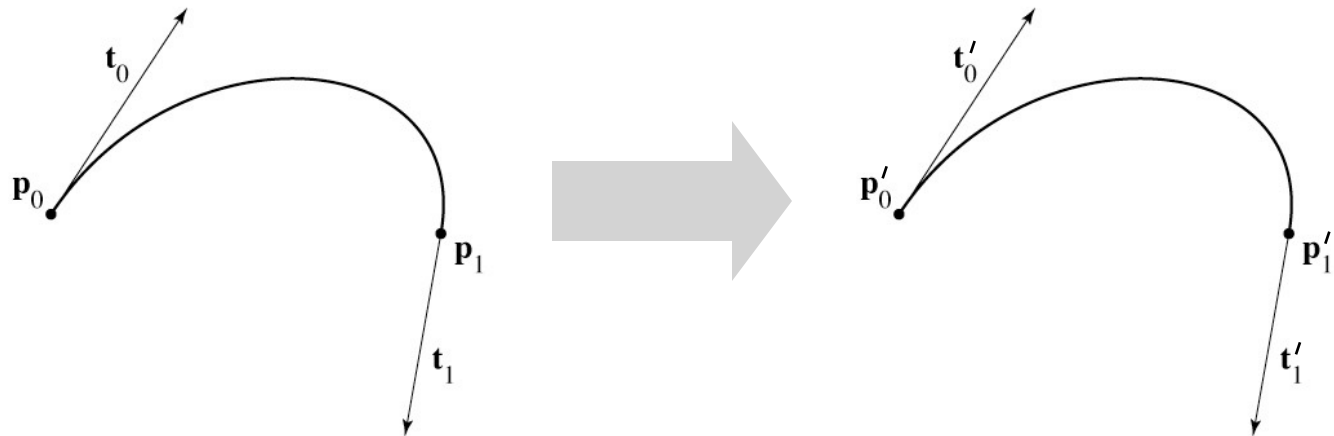
Affine invariance

- Transforming the control points is the same as transforming the curve
 - true for all commonly used splines
 - extremely convenient in practice...



Affine invariance

- Basis functions associated with points should always sum to 1



$$\begin{aligned}\mathbf{p}(t) &= b_0\mathbf{p}_0 + b_1\mathbf{p}_1 + b_2\mathbf{v}_0 + b_3\mathbf{v}_1 \\ \mathbf{p}'(t) &= b_0(\mathbf{p}_0 + \mathbf{u}) + b_1(\mathbf{p}_1 + \mathbf{u}) + b_2\mathbf{v}_0 + b_3\mathbf{v}_1 \\ &= b_0\mathbf{p}_0 + b_1\mathbf{p}_1 + b_2\mathbf{v}_0 + b_3\mathbf{v}_1 + (b_0 + b_1)\mathbf{u} \\ &= \mathbf{p}(t) + \mathbf{u}\end{aligned}$$

Spline Curves

Chaining spline segments

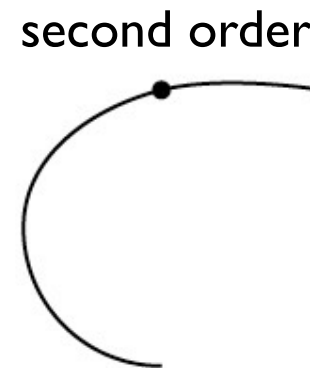
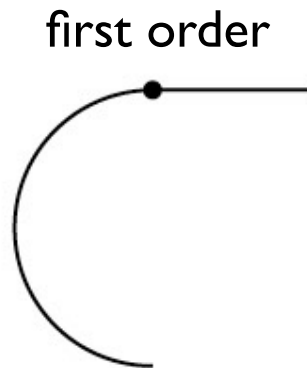
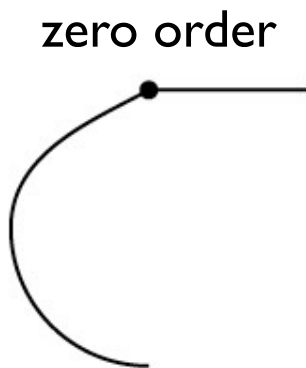
- Can only do so much with a single polynomial
- Can use these functions as segments of a longer curve
 - curve from $t = 0$ to $t = 1$ defined by first segment
 - curve from $t = 1$ to $t = 2$ defined by second segment

$$\mathbf{f}(t) = \mathbf{f}_i(t - i) \quad \text{for } i \leq t \leq i + 1$$

- To avoid discontinuity, match derivatives at junctions
 - this produces a C^1 curve

Continuity

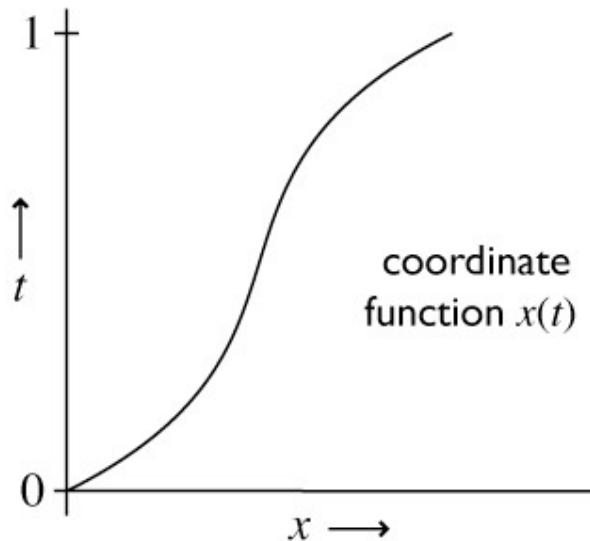
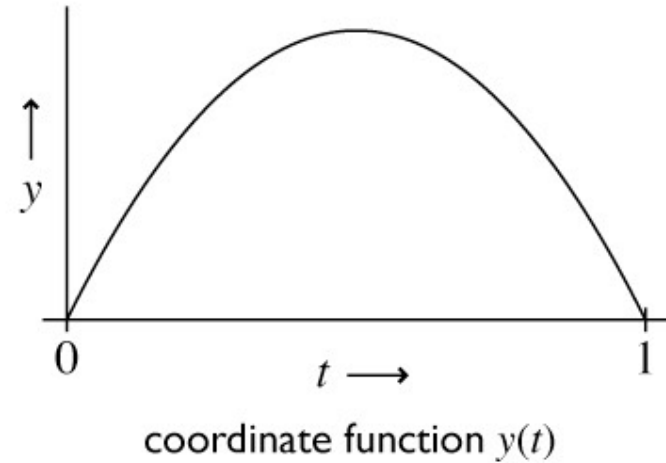
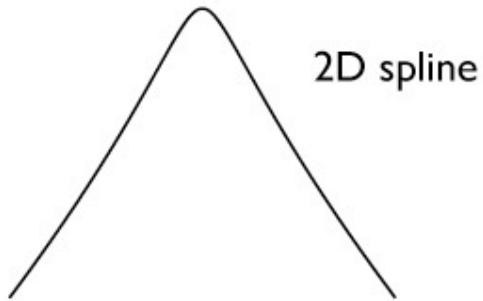
- Smoothness can be described by degree of continuity
 - zero-order (C^0): position matches from both sides
 - first-order (C^1): tangent matches from both sides
 - second-order (C^2): curvature matches from both sides
 - G^n vs. C^n



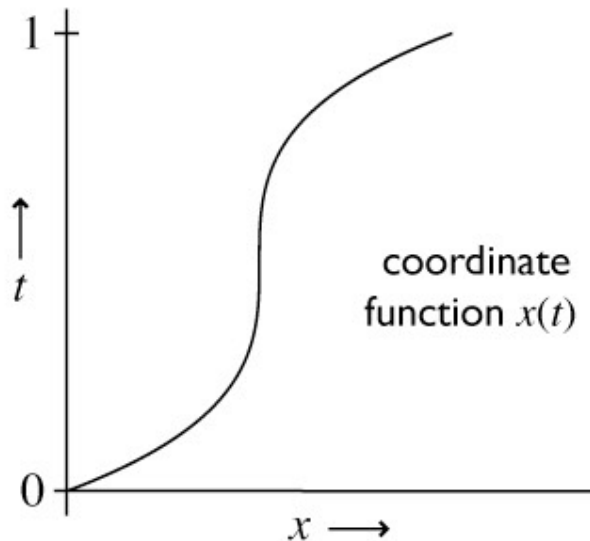
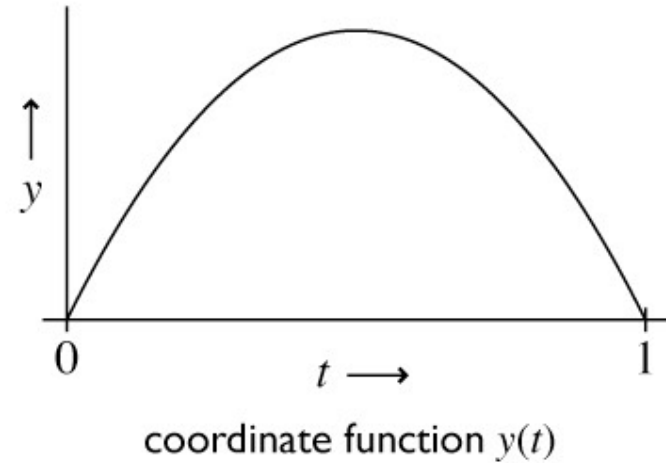
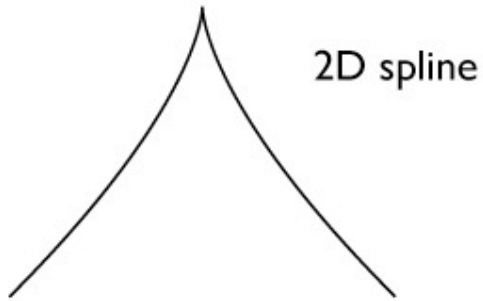
Continuity

- Parametric continuity (C) of spline is continuity of coordinate functions
- Geometric continuity (G) is continuity of the curve itself
- Neither form of continuity is guaranteed by the other
 - Can be C^1 but not G^1 when $\mathbf{p}(t)$ comes to a halt (next slide)
 - Can be G^1 but not C^1 when the tangent vector changes length abruptly

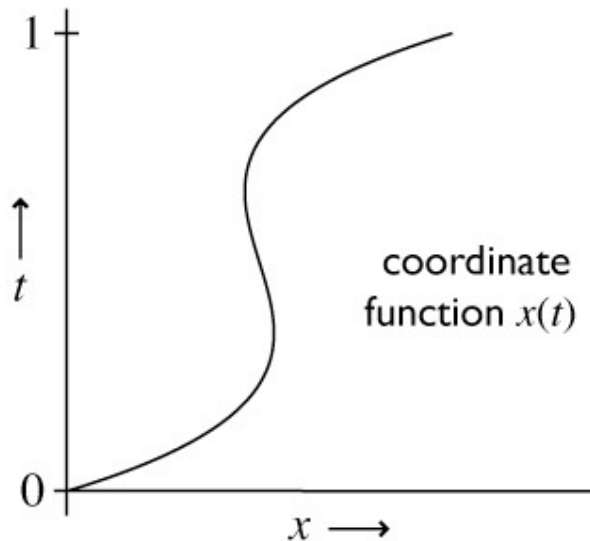
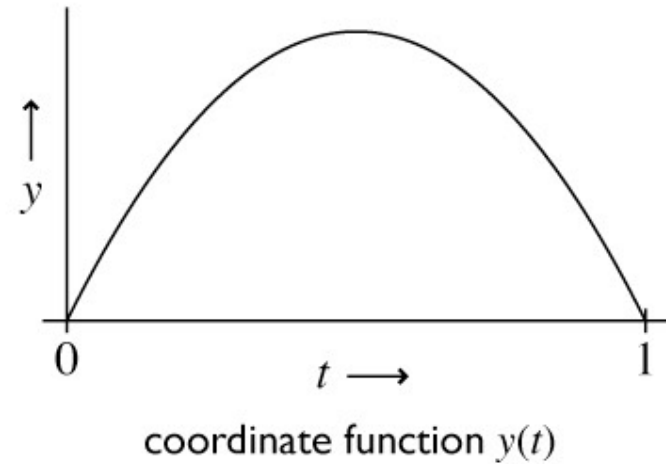
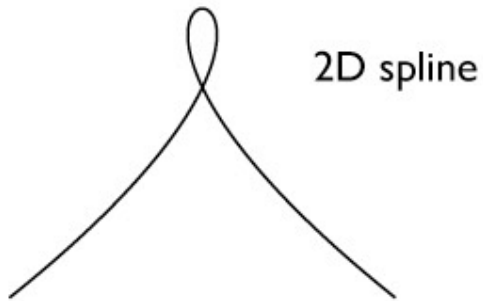
Geometric vs. parametric continuity



Geometric vs. parametric continuity

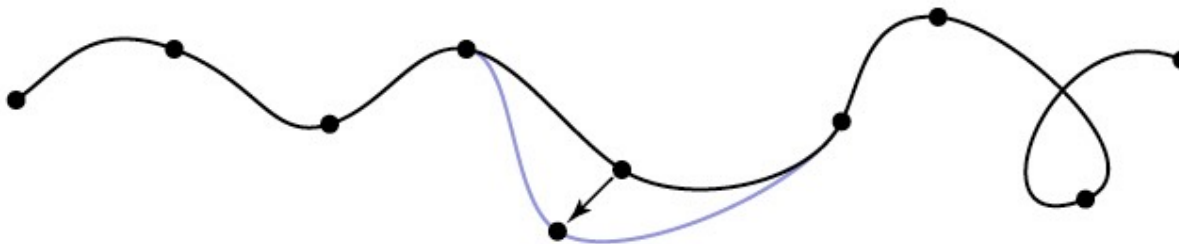


Geometric vs. parametric continuity



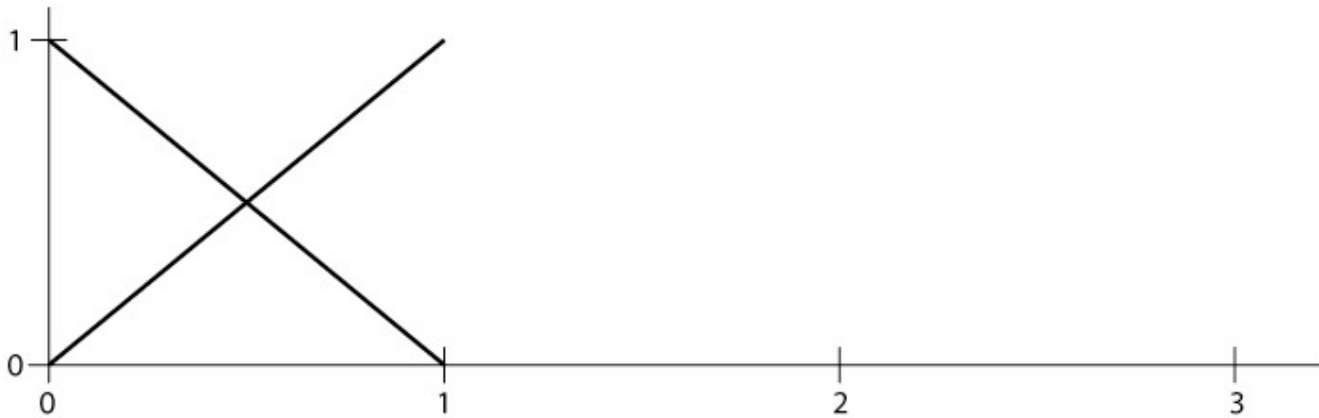
Control

- Local control
 - changing control point only affects a limited part of spline
 - without this, splines are very difficult to use
 - many likely formulations lack this
 - natural spline
 - polynomial fits



Trivial example: piecewise linear

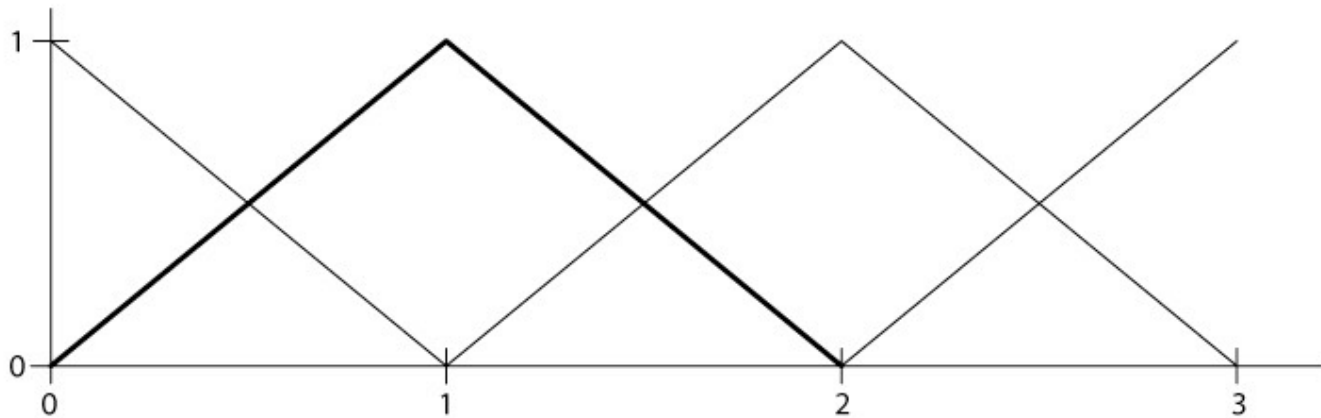
- Basis function formulation: “function times point”
 - basis functions: contribution of each point as t changes



- can think of them as blending functions glued together
- this is just like a reconstruction filter!

Trivial example: piecewise linear

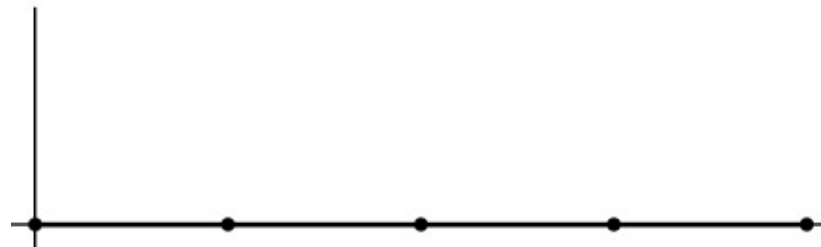
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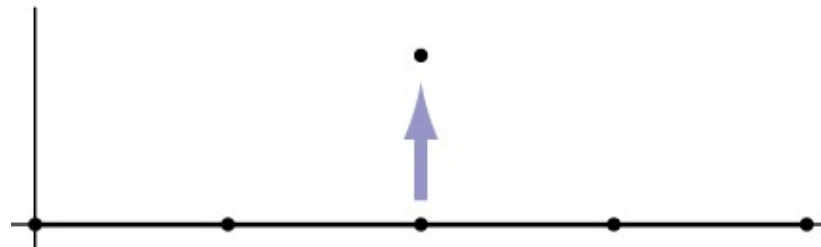
Seeing the basis functions

- Basis functions of a spline are revealed by how the curve changes in response to a change in one control
 - to get a graph of the basis function, start with the curve laid out in a straight, constant-speed line
 - what are $x(t)$ and $y(t)$?
 - then move one control straight up



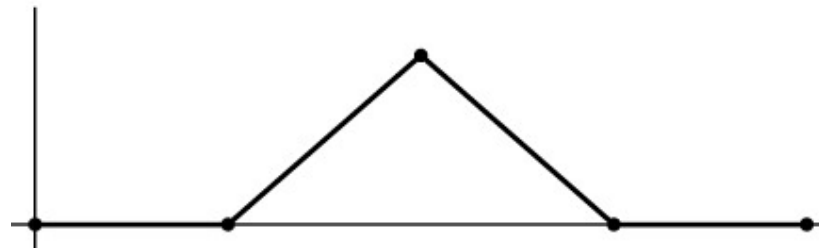
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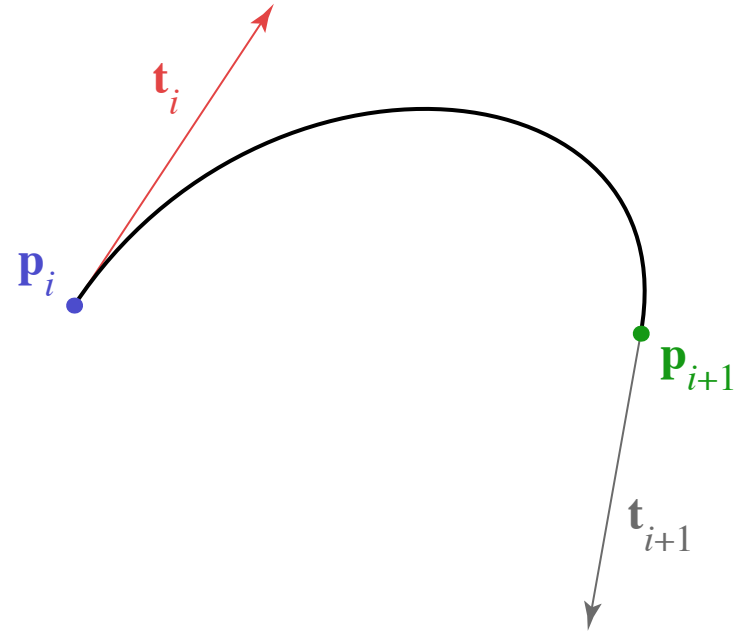
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Hermite splines

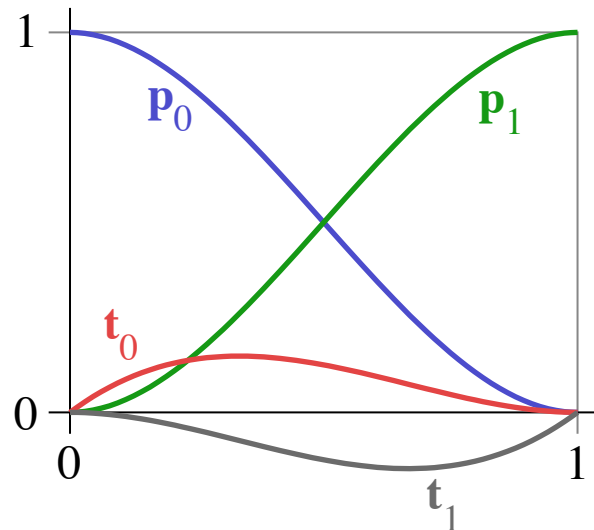
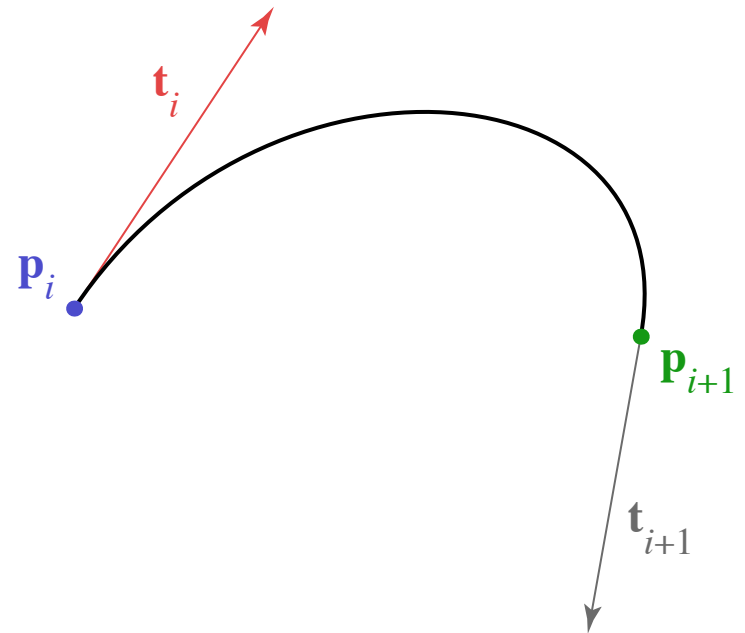
- Constraints are endpoints and endpoint tangents



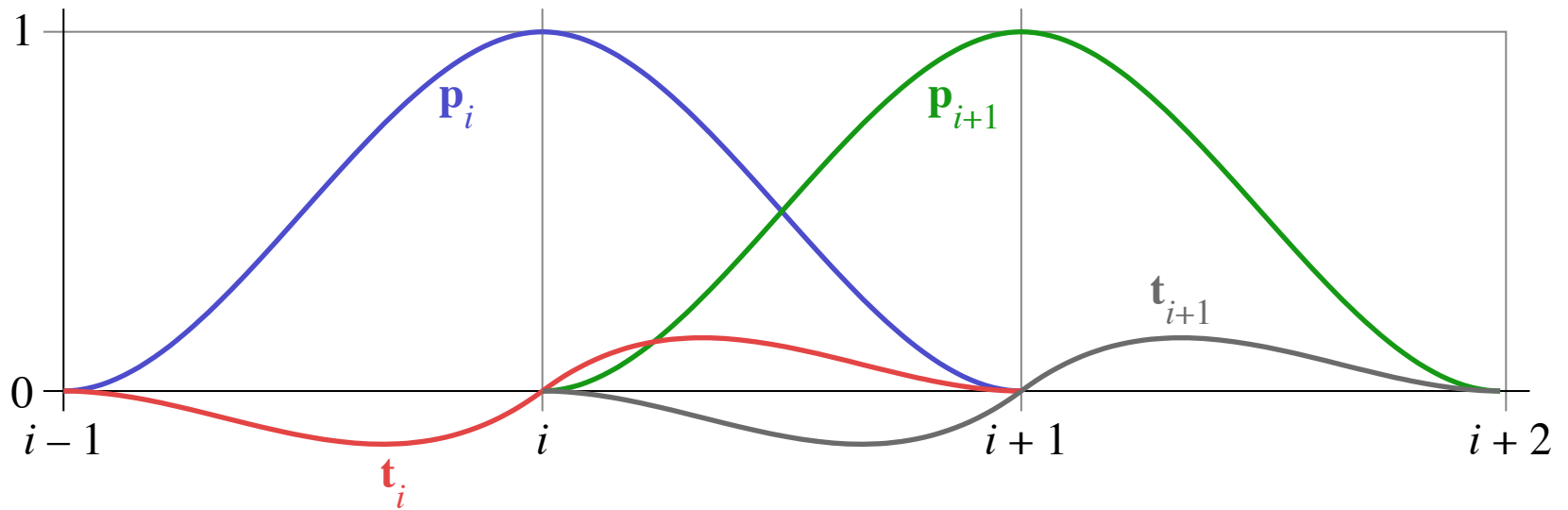
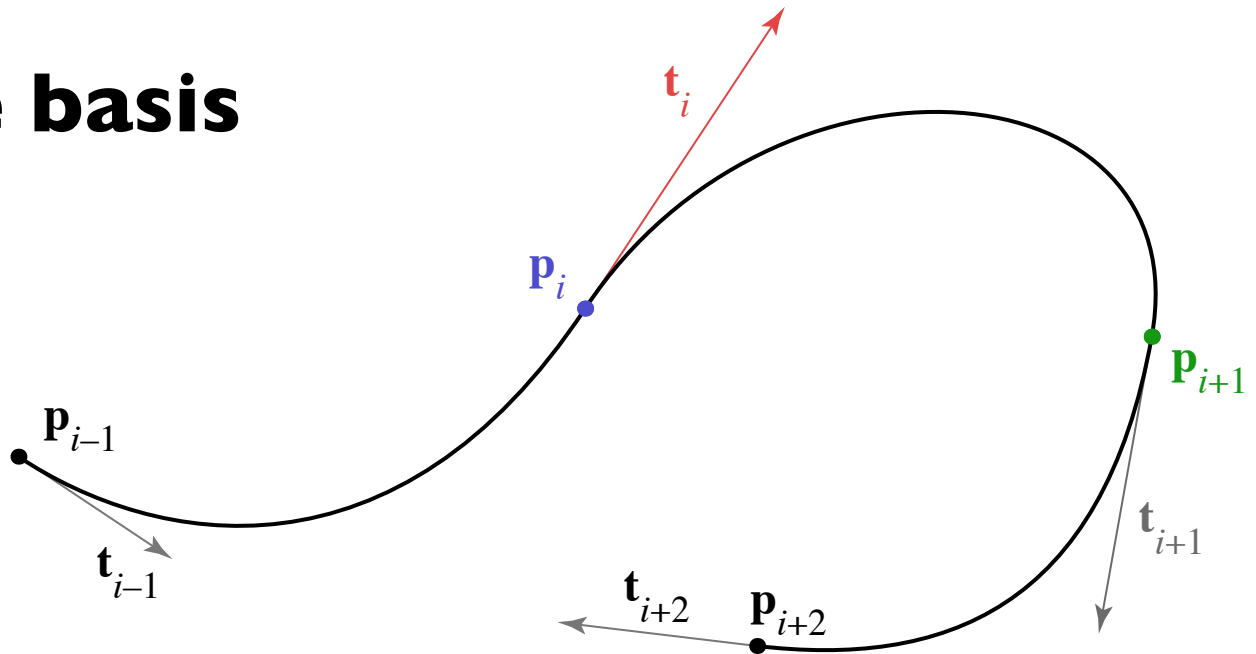
$$\mathbf{f}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 2 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}'_0 \\ \mathbf{p}'_1 \end{bmatrix}$$

Hermite basis

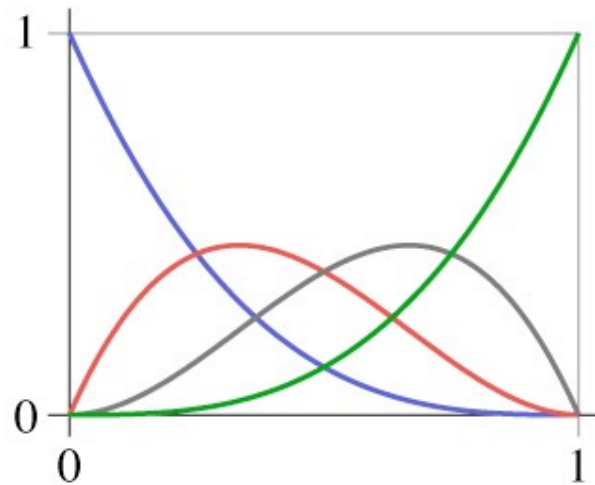
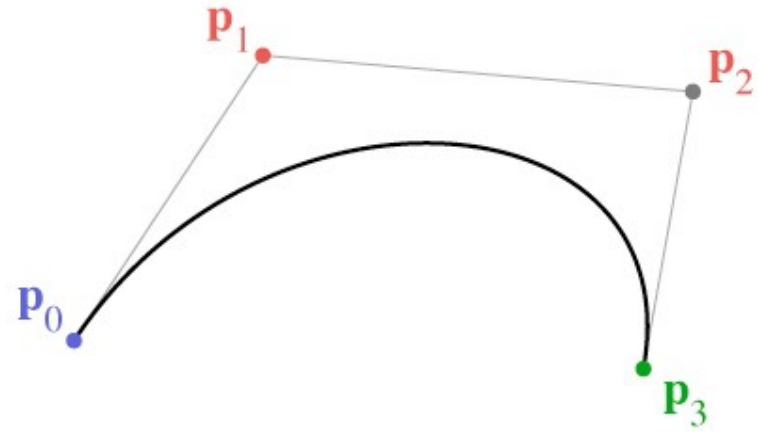
Hermite basis



Hermite basis



Bézier basis

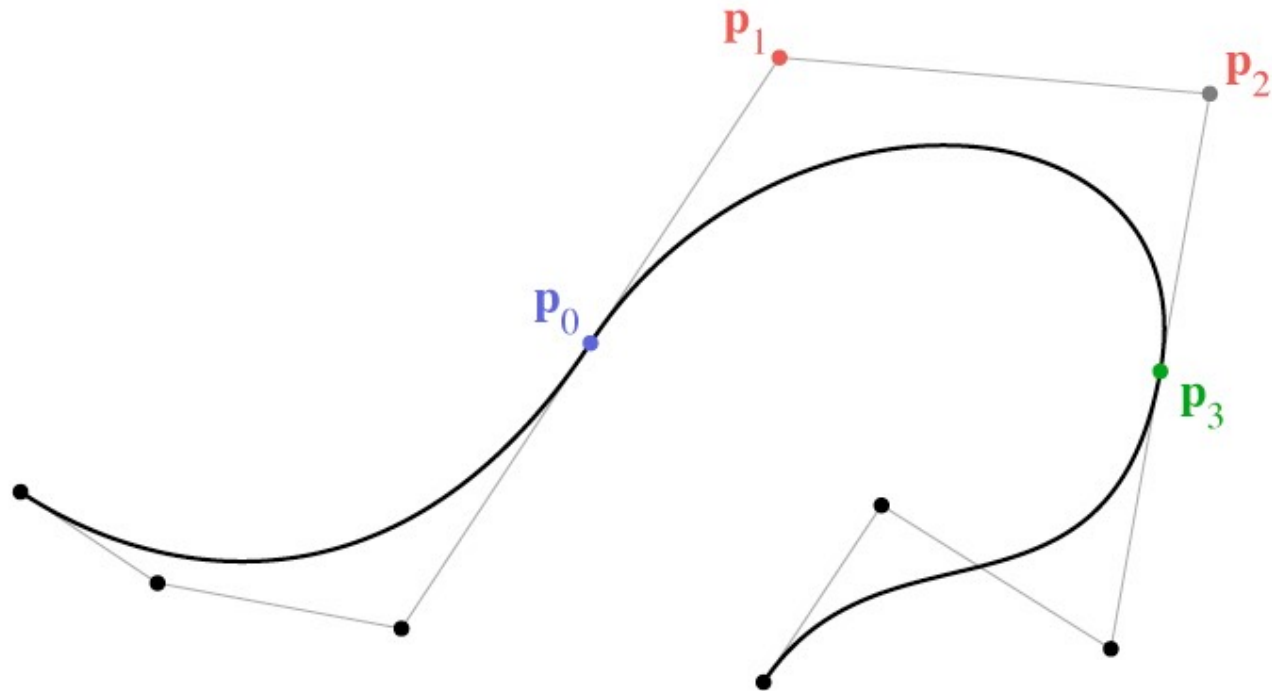


Chaining Bézier splines

- No continuity built in
- Achieve C^1 using collinear control points

Chaining Bézier splines

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- Achieve C^1 using collinear control points

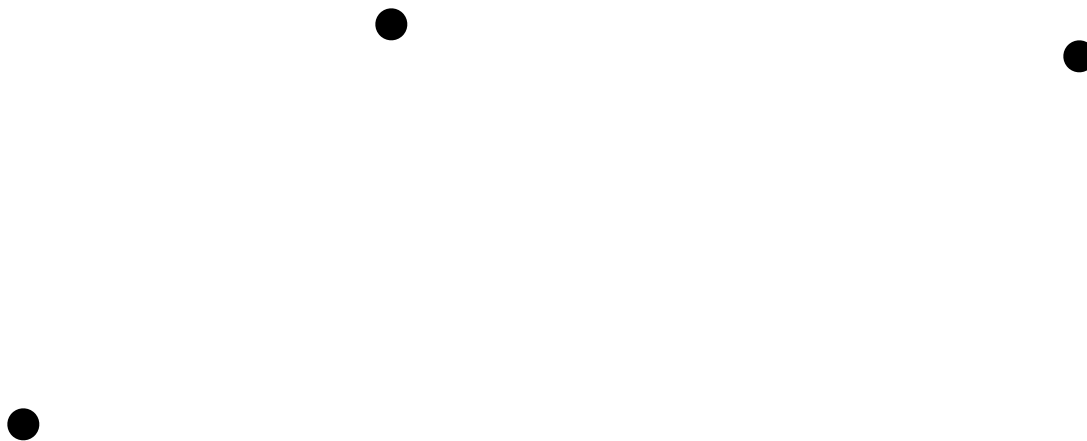


Chaining spline segments

- Hermite curves are convenient because they can be made long easily
- Bézier curves are convenient because their controls are all points
 - but it is fussy to maintain continuity constraints
 - and they interpolate every 3rd point, which is a little odd
- We derived Bézier from Hermite by defining tangents from control points
 - a similar construction leads to the interpolating *Catmull-Rom* spline

Hermite to Catmull-Rom

- Have not yet seen any interpolating splines
- Would like to define tangents automatically
 - use adjacent control points

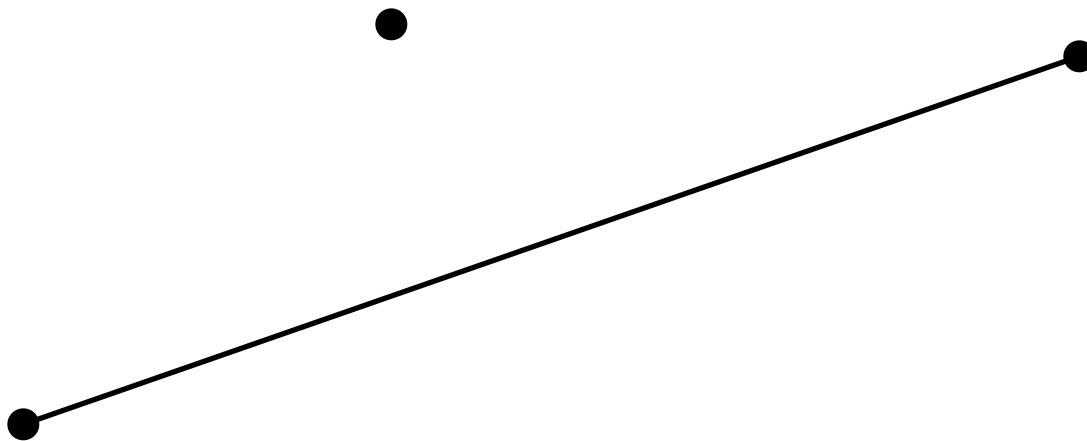


- end tangents: extra points or zero



Hermite to Catmull-Rom

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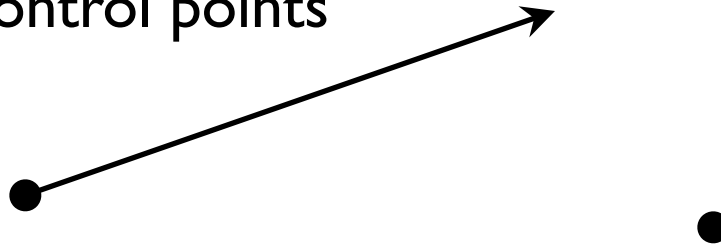


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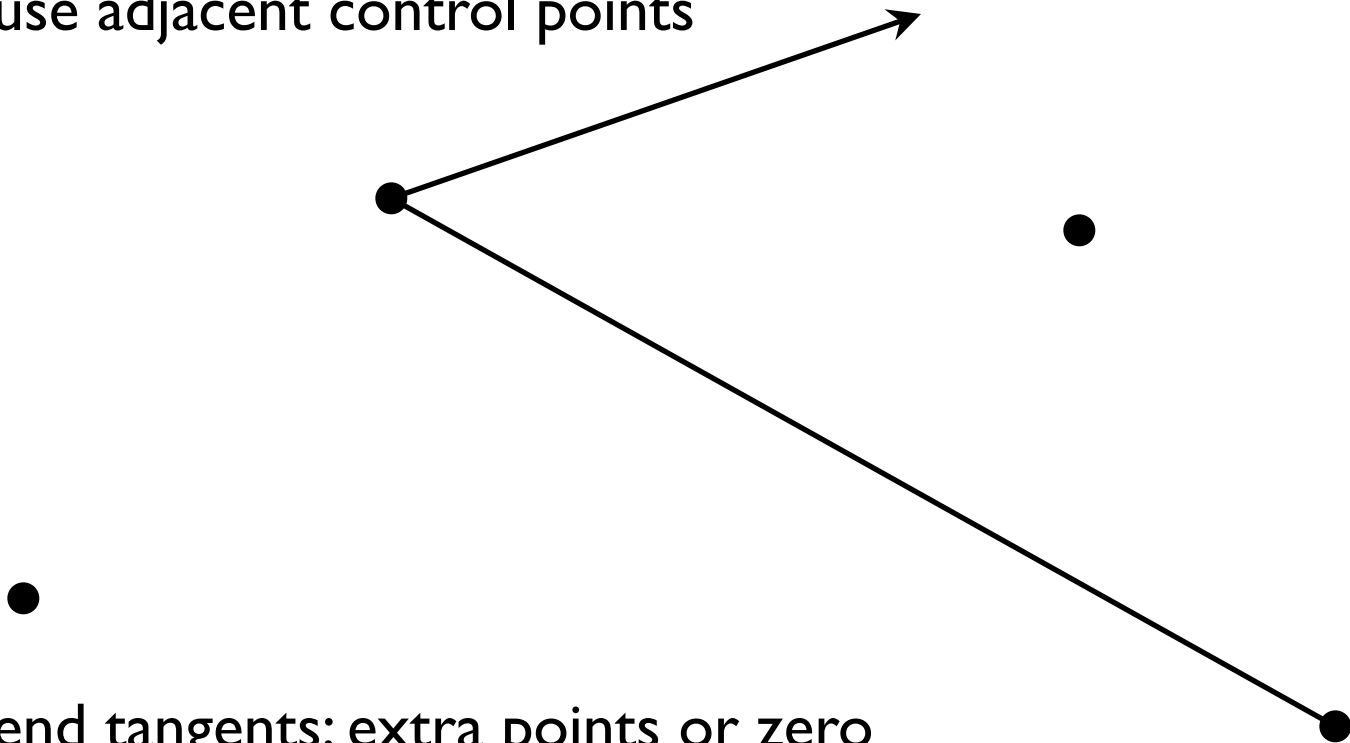


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Hermite to Catmull-Rom

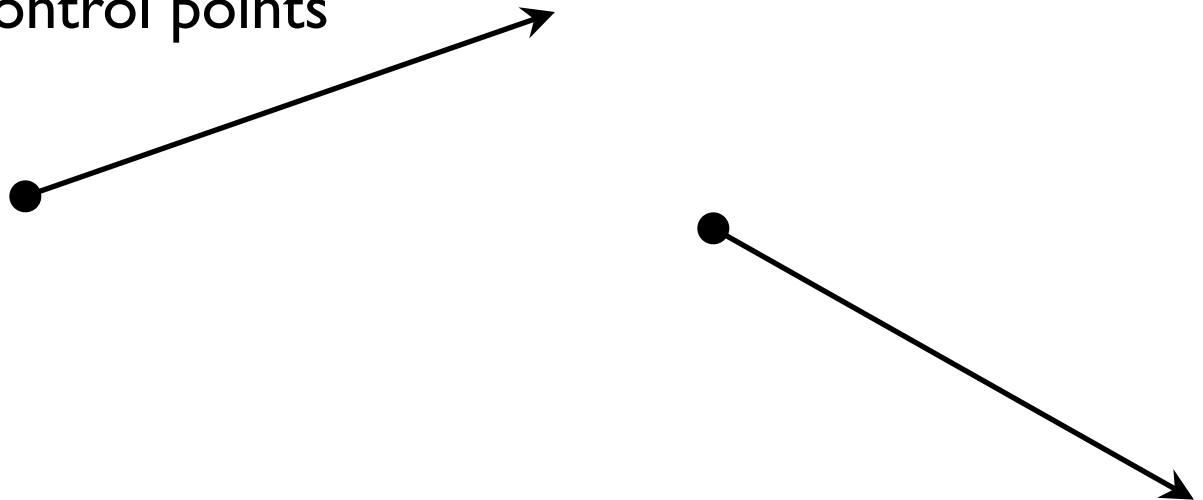
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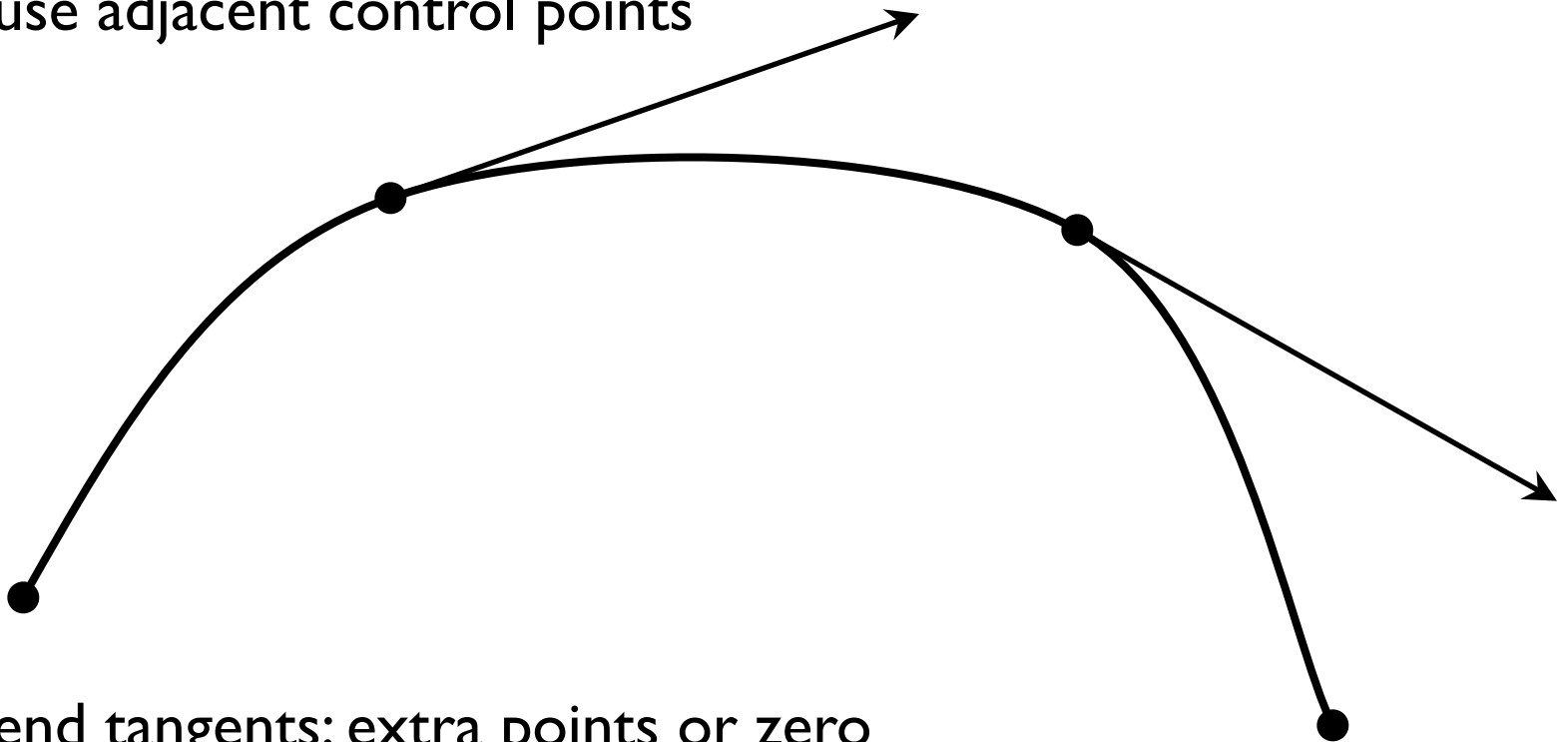


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Hermite to Catmull-Rom

- Have not yet seen any interpolating splines
- Would like to define tangents automatically
 - use adjacent control points



- end tangents: extra points or zero

Hermite to Catmull-Rom

- Tangents are $(\mathbf{p}_{k+1} - \mathbf{p}_{k-1}) / 2$
 - scaling based on same argument about collinear case

$$\mathbf{p}_0 = \mathbf{q}_k$$

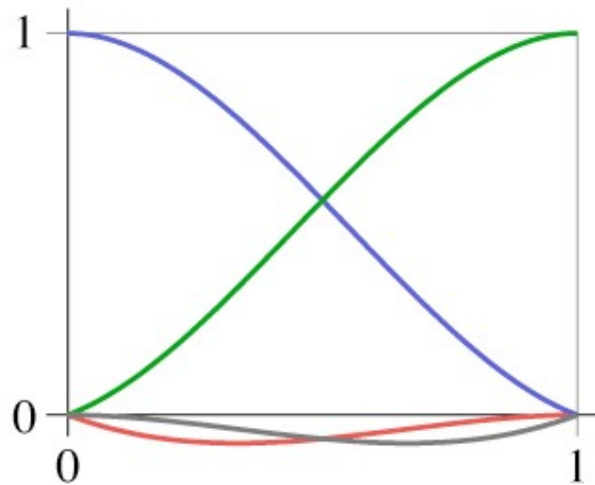
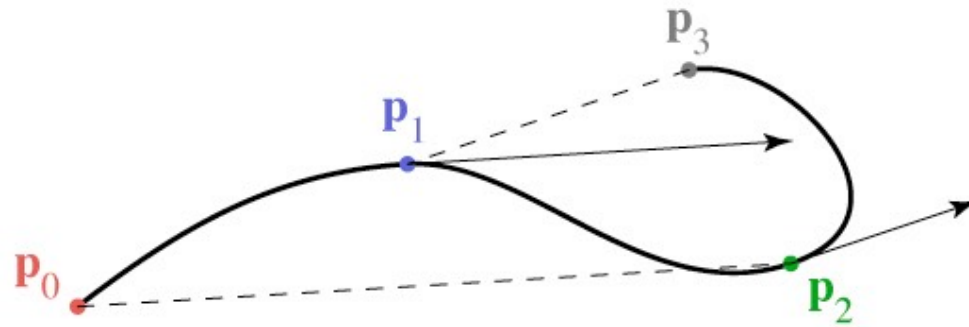
$$\mathbf{p}_1 = \mathbf{q}_{k+1}$$

$$\mathbf{v}_0 = 0.5(\mathbf{q}_{k+1} - \mathbf{q}_{k-1})$$

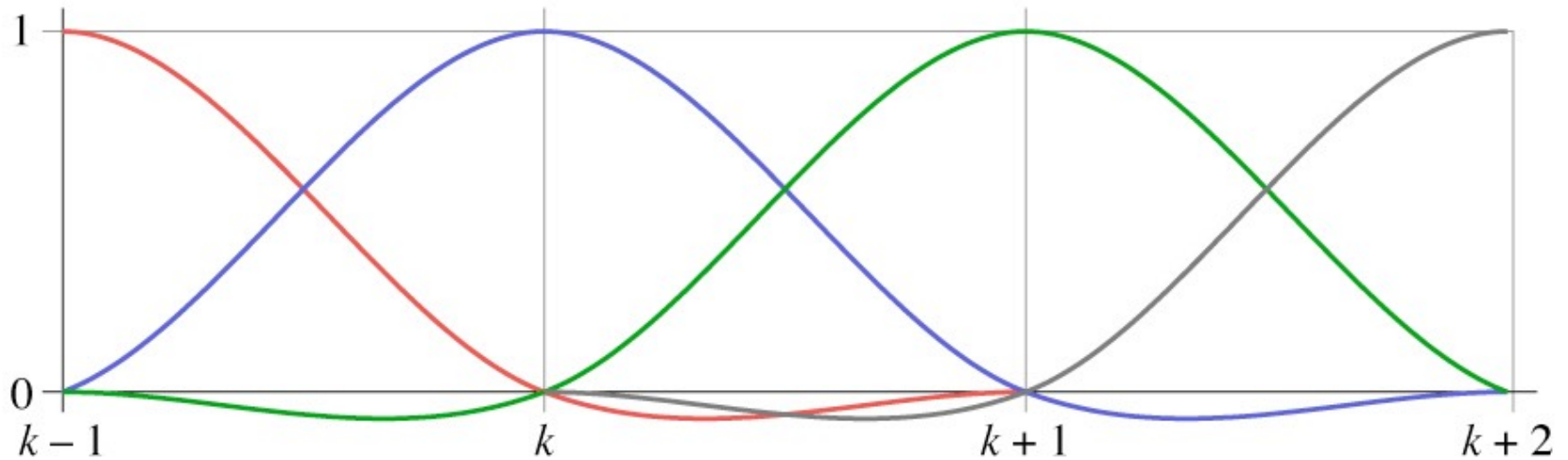
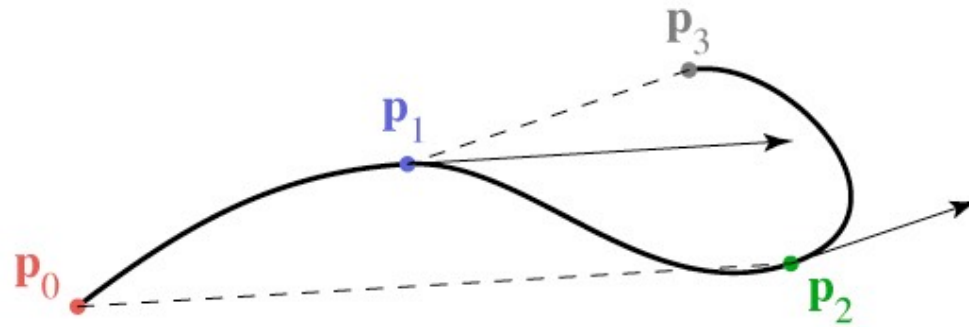
$$\mathbf{v}_1 = 0.5(\mathbf{q}_{k+2} - \mathbf{q}_K)$$

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -.5 & 0 & .5 & 0 \\ 0 & -.5 & 0 & .5 \end{bmatrix} \begin{bmatrix} \mathbf{q}_{k-1} \\ \mathbf{q}_k \\ \mathbf{q}_{k+1} \\ \mathbf{q}_{k+2} \end{bmatrix}$$

Catmull-Rom basis



Catmull-Rom basis



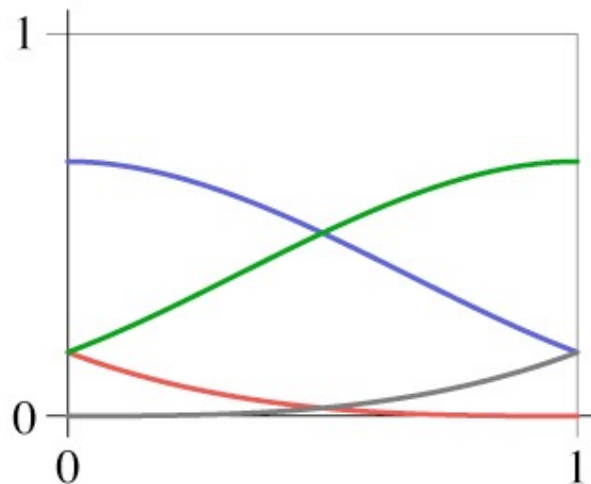
Catmull-Rom splines

- Our first example of an interpolating spline
- Like Bézier, equivalent to Hermite
 - in fact, all splines of this form are equivalent
- First example of a spline based on just a control point sequence
- Does not have convex hull property

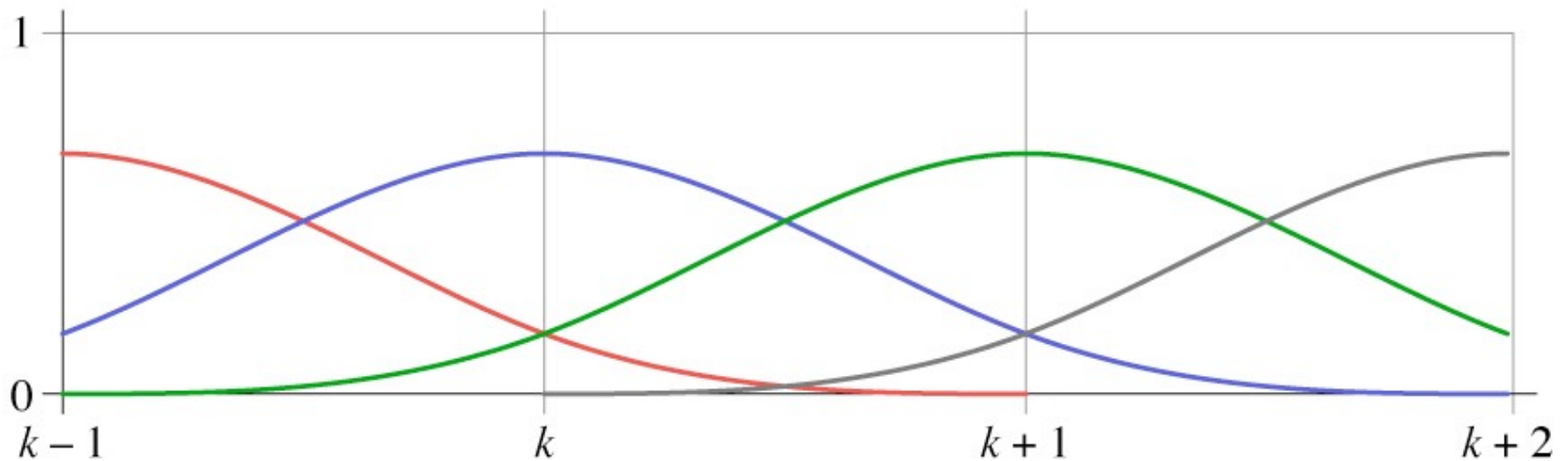
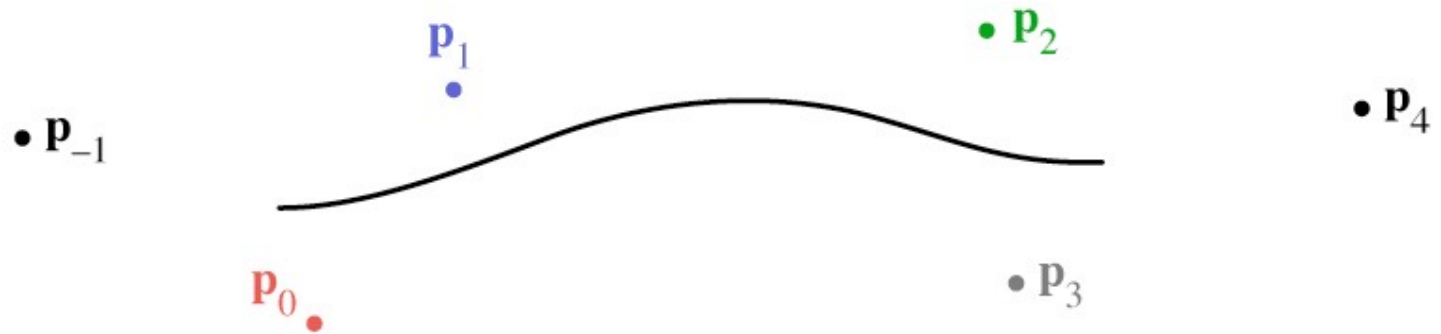
B-splines

- We may want more continuity than C^1
- We may not need an interpolating spline
- B-splines are a clean, flexible way of making long splines with arbitrary order of continuity
- Various ways to think of construction
 - a simple one is convolution
 - relationship to sampling and reconstruction

Cubic B-spline basis



Cubic B-spline basis



Deriving the B-Spline

- Approached from a different tack than Hermite-style constraints
 - Want a cubic spline; therefore 4 active control points
 - Want C^2 continuity
 - Turns out that is enough to determine everything

Efficient construction of any B-spline

- B-splines defined for all orders
 - order d : degree $d - 1$
 - order d : d points contribute to value
- One definition: Cox-deBoor recurrence

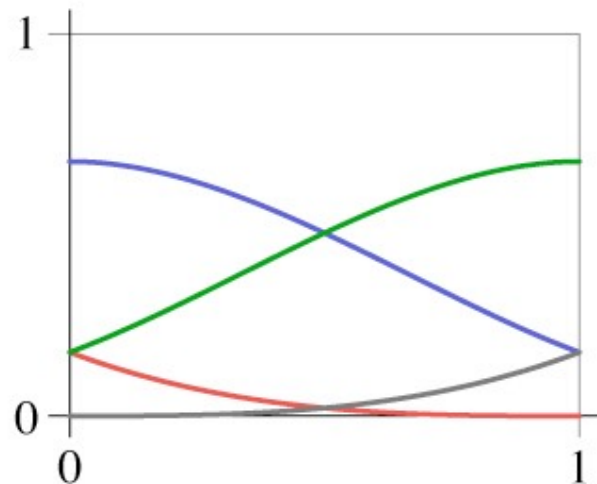
$$b_1 = \begin{cases} 1 & 0 \leq u < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_d = \frac{t}{d-1} b_{d-1}(t) + \frac{d-t}{d-1} b_{d-1}(t-1)$$

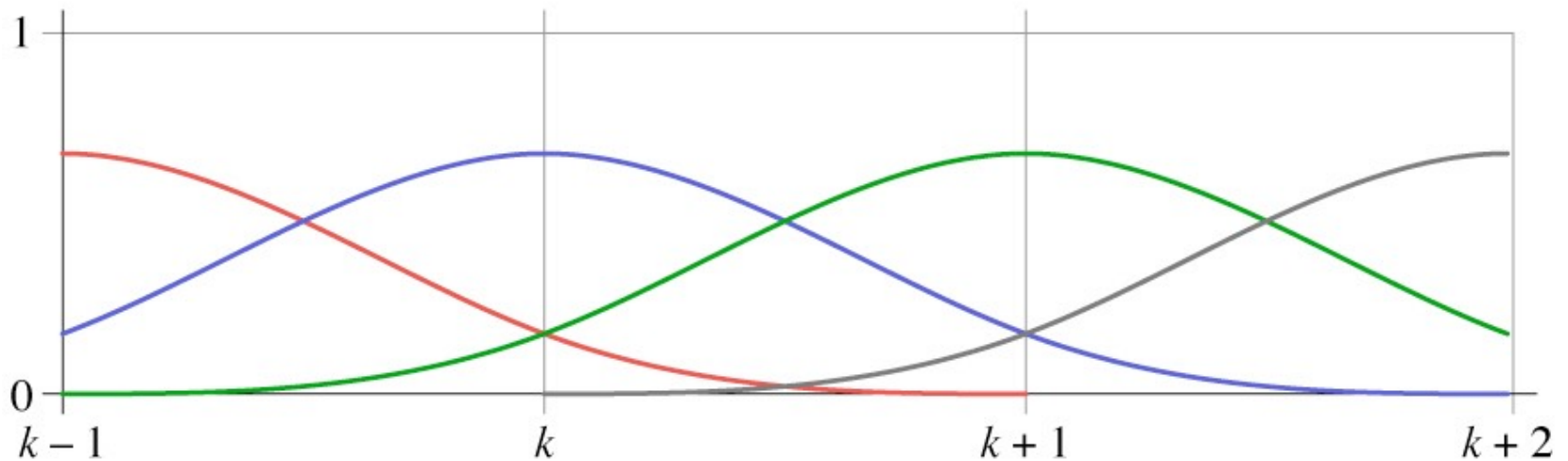
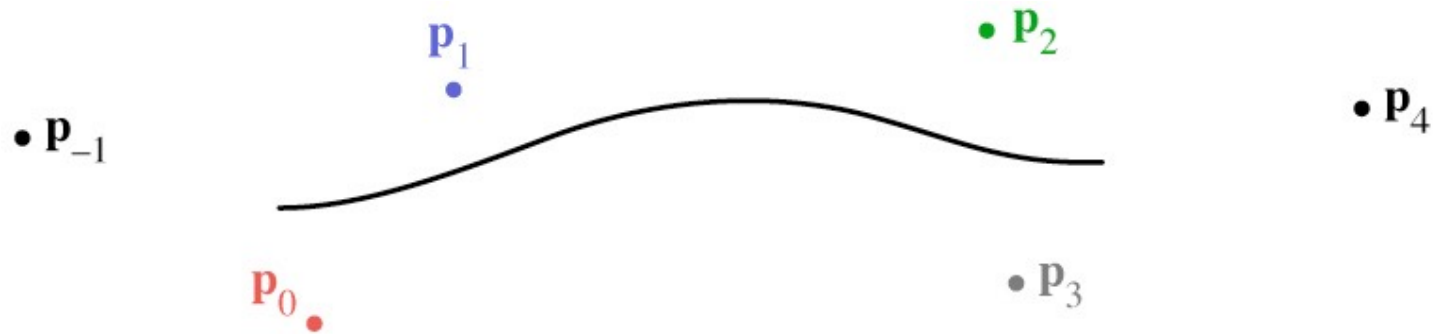
Cubic B-spline matrix

$$\mathbf{f}_i(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \\ \mathbf{p}_{i+2} \end{bmatrix}$$

Cubic B-spline basis



Cubic B-spline basis



Refinement and Evaluation

Converting spline representations

- All the splines we have seen so far are equivalent
 - all represented by geometry matrices

$$\mathbf{p}_S(t) = T(t)M_S P_S$$

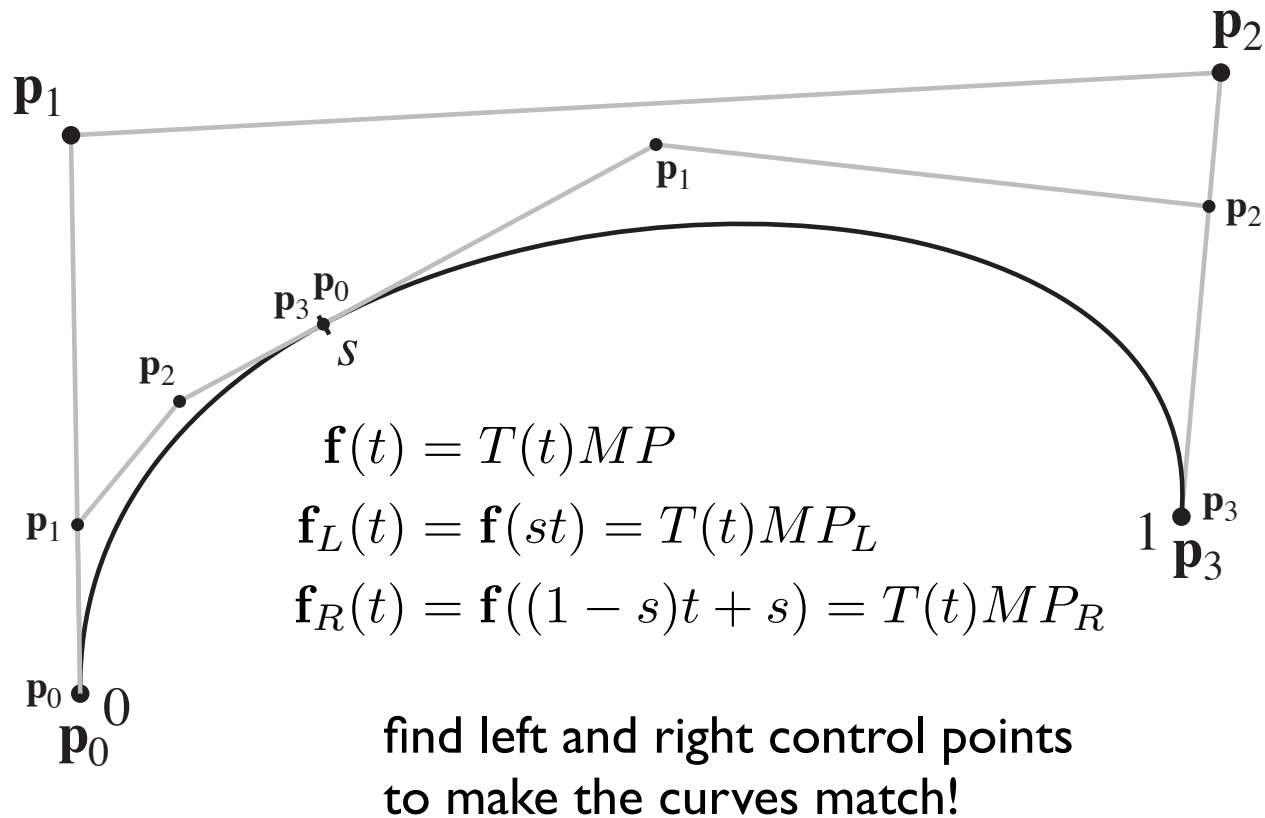
- where S represents the type of spline
 - therefore the control points may be transformed from one type to another using matrix multiplication

$$P_1 = M_1^{-1} M_2 P_2$$

$$\begin{aligned}\mathbf{p}_1(t) &= T(t)M_1(M_1^{-1}M_2P_2) \\ &= T(t)M_2P_2 = \mathbf{p}_2(t)\end{aligned}$$

Refinement of splines

- May want to add more control to a curve
- Can add control by splitting a segment into two



Refinement math

$$\begin{aligned}\mathbf{f}_L(t) &= T(st)MP = T(t)S_LMP \\ &= T(t)M(M^{-1}S_LMP) \\ &= T(t)MP_L\end{aligned}$$

$$P_L = M^{-1}S_LMP$$

$$P_R = M^{-1}S_RMP$$

$$S_L = \begin{bmatrix} s^3 & & & \\ & s^2 & & \\ & & s & \\ & & & 1 \end{bmatrix}$$
$$S_R = \begin{bmatrix} s^3 & & & \\ 3s^2(1-s) & s^2 & & \\ 3s(1-s)^2 & 2s(1-s) & s & \\ (1-s)^3 & (1-s)^2 & (1-s) & 1 \end{bmatrix}$$

Other types of B-splines

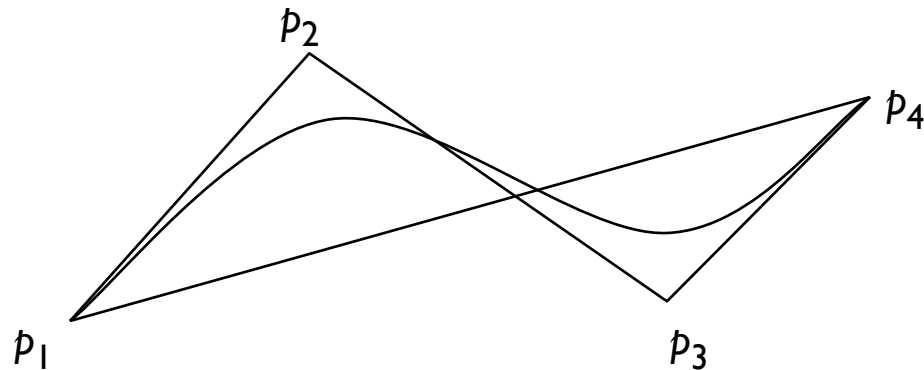
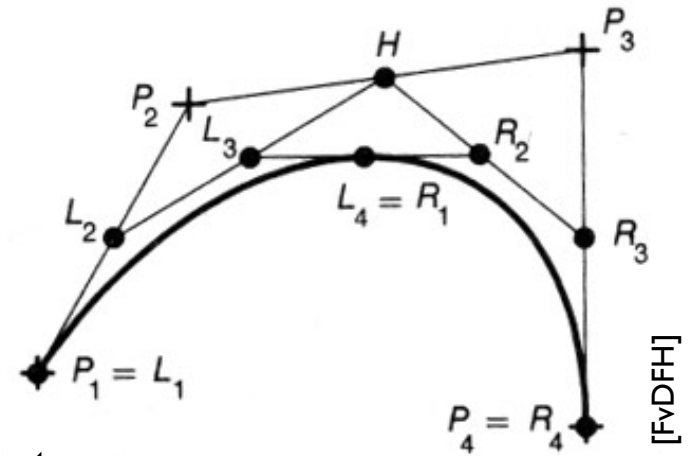
- Nonuniform B-splines
 - discontinuities not evenly spaced
 - allows control over continuity or interpolation at certain points
 - e.g. interpolate endpoints (commonly used case)
- Nonuniform Rational B-splines (NURBS)
 - ratios of nonuniform B-splines: $x(t) / w(t)$; $y(t) / w(t)$
 - key properties:
 - invariance under perspective as well as affine
 - ability to represent conic sections exactly

Evaluating splines for display

- Need to generate a list of line segments to draw
 - generate efficiently
 - use as few as possible
 - guarantee approximation accuracy
- Approaches
 - recursive subdivision (easy to do adaptively)
 - uniform sampling (easy to do efficiently)

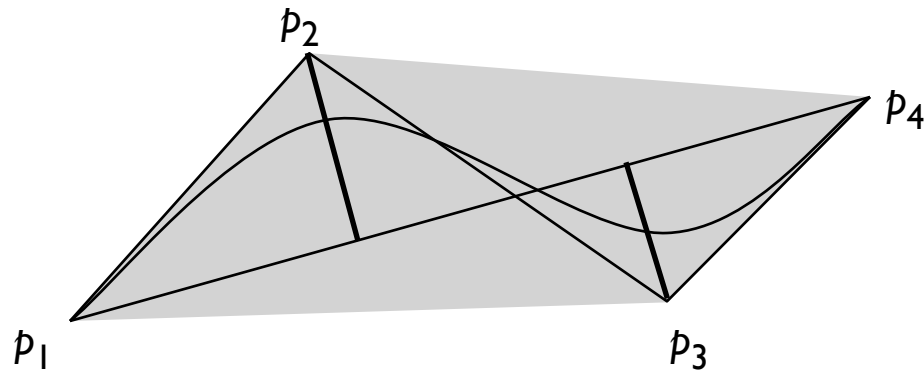
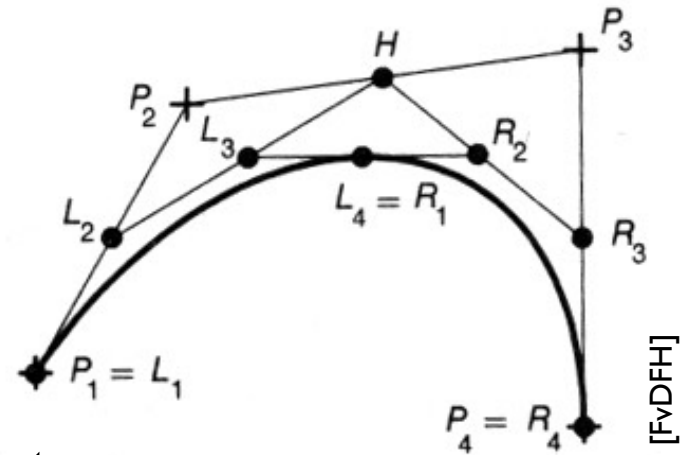
Evaluating by subdivision

- Recursively split spline
 - stop when polygon is within epsilon of curve
 - Termination criteria
 - distance between control points
 - distance of control points from line
 - angles in control polygon
- 
- The diagram shows a curved line representing a spline. A straight line segment, the control polygon, is drawn above the curve. The endpoints of the control polygon are labeled P_1 and P_2 . A point L_2 is marked on the control polygon between P_1 and P_2 . A dashed line segment connects P_1 to L_2 , and another dashed line segment connects L_2 to P_2 . The label $P_1 =$ is partially visible at the bottom right.



Evaluating by subdivision

- Recursively split spline
 - stop when polygon is within epsilon of curve
- Termination criteria
 - distance between control points
 - distance of control points from line
 - angles in control polygon



Evaluating with uniform spacing

- Forward differencing
 - efficiently generate points for uniformly spaced t values
 - evaluate polynomials using repeated differences

Surfaces built from curves

- Parametric spline surfaces
 - extrusions
 - surfaces of revolution
 - generalized cylinders
 - spline patches
- Pause for differential geometry primer...
 - plane and space curves, tangent vectors
 - parametric surfaces, isolines, tangent vectors, normals

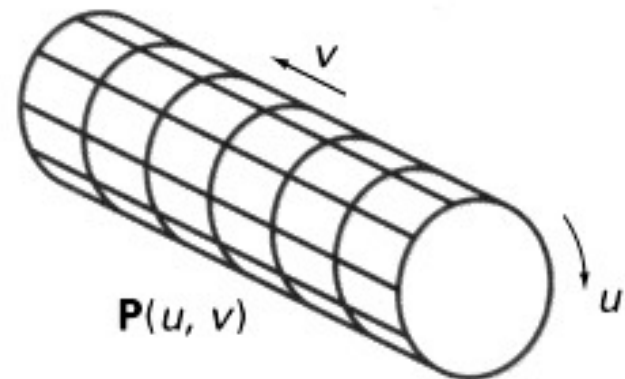
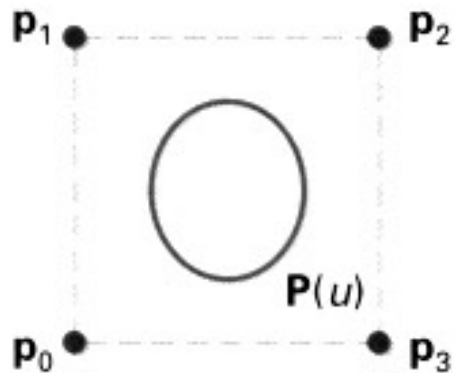
From curves to surfaces

- So far have discussed spline curves in 2D
 - it turns out that this already provides of the mathematical machinery for several ways of building curved surfaces
- Building surfaces from 2D curves
 - extrusions and surfaces of revolution
- Building surfaces from 2D and 3D curves
 - generalized swept surfaces
- Building surfaces from spline patches
 - generalizing spline curves to spline patches
- Also to think about: generating triangles

Extrusions

- Given a spline curve $C \in \mathbb{R}^2$, define $S \in \mathbb{R}^3$ by
$$S = C \times [a, b]$$
- This produces a “tube” with the given cross section
 - Circle: cylinder; “L”: shelf bracket; “I”: I beam
- It is parameterized by the spline t and the interval $[a, b]$

$$s(t, s) = [c_x(t), c_y(t), s]^T$$

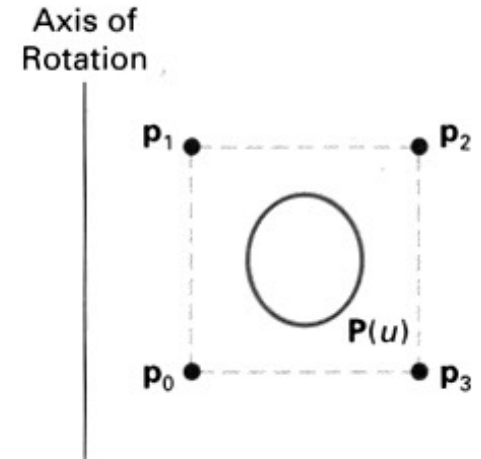


Surfaces of revolution

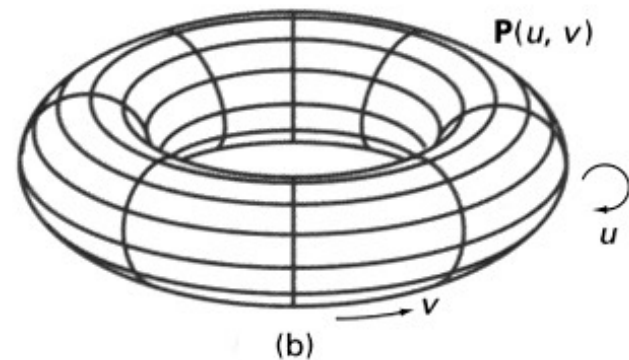
- Take a 2D curve and spin it around an axis
- Given curve $\mathbf{c}(t)$ in the plane, the surface is defined easily in cylindrical coordinates:

$$\mathbf{s}(t, s) = (r, \phi, z) = (c_x(t), s, c_y(t))$$

- the torus is an example in which the curve $\mathbf{c}$ is a circle



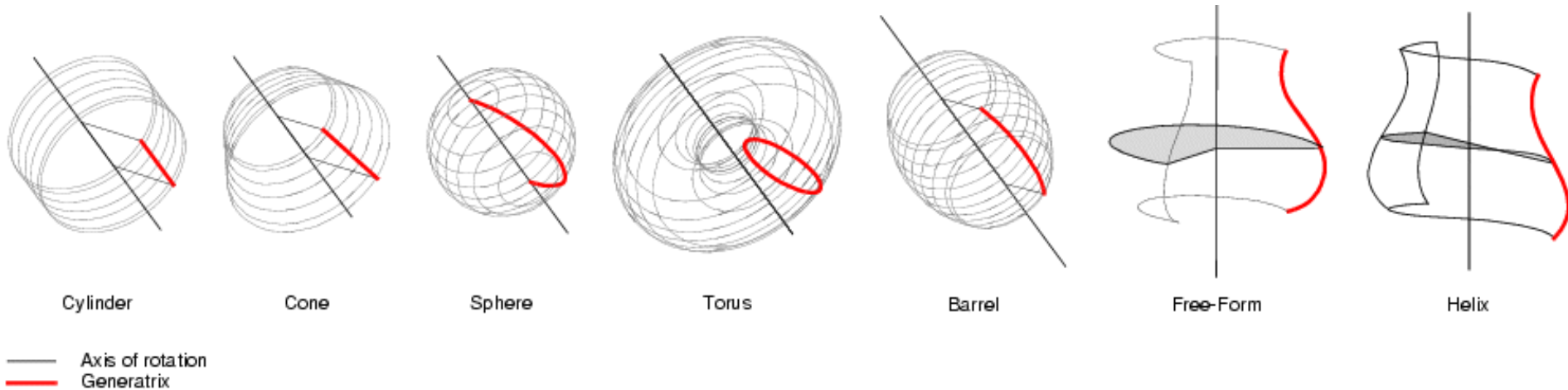
(a)



(b)

Surfaces of revolution

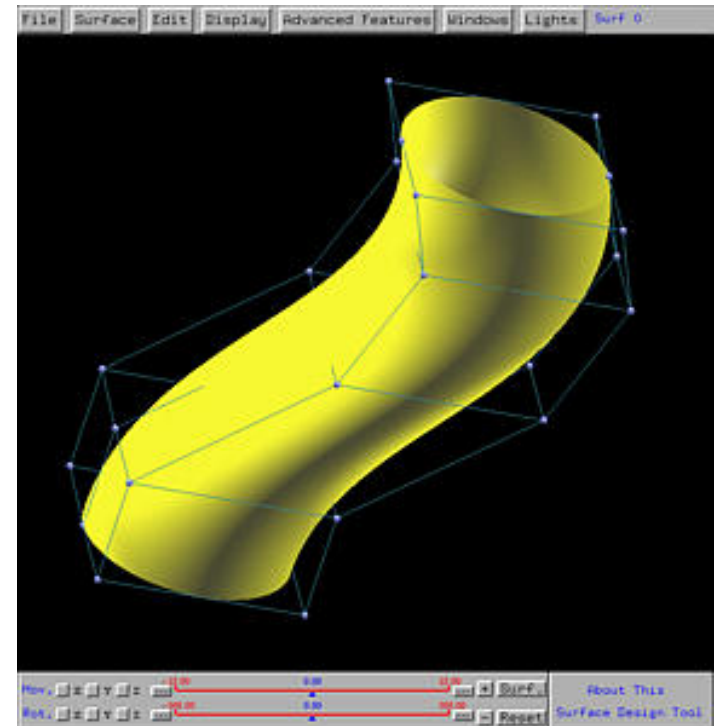
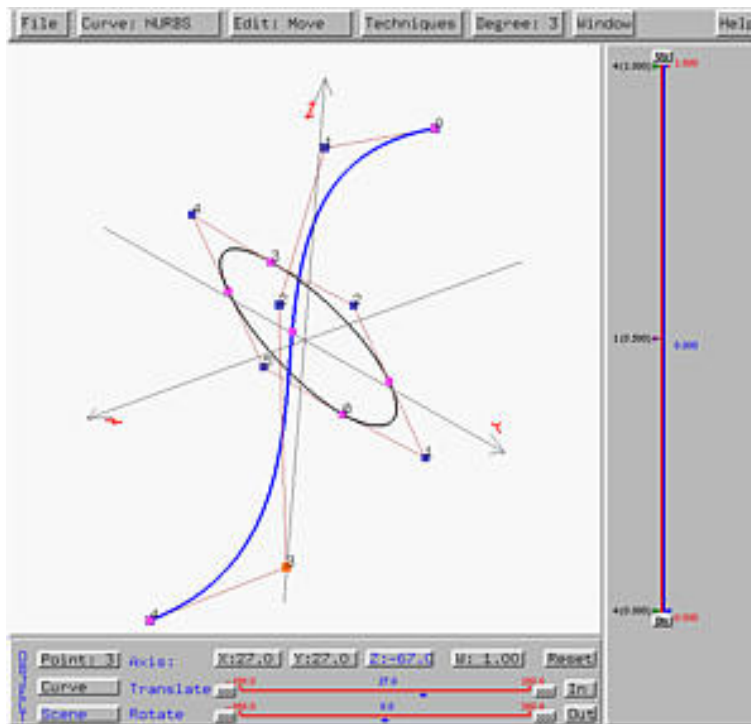
- More examples...



[CATIA documentation]

Swept surfaces

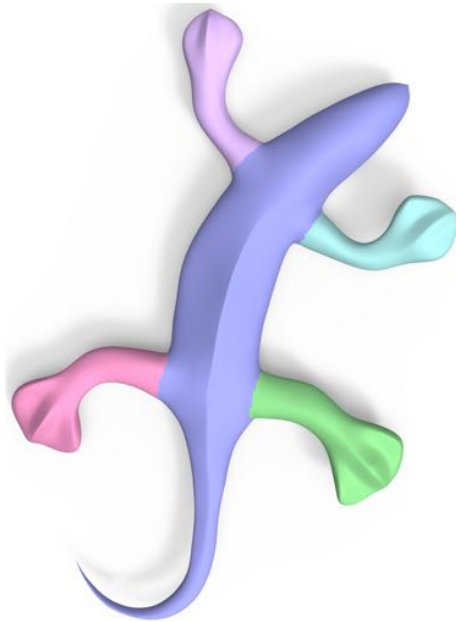
- Surface defined by a *cross section* moving along a *spine*
- Simple version: a single 3D curve for spine and a single 2D curve for the cross section



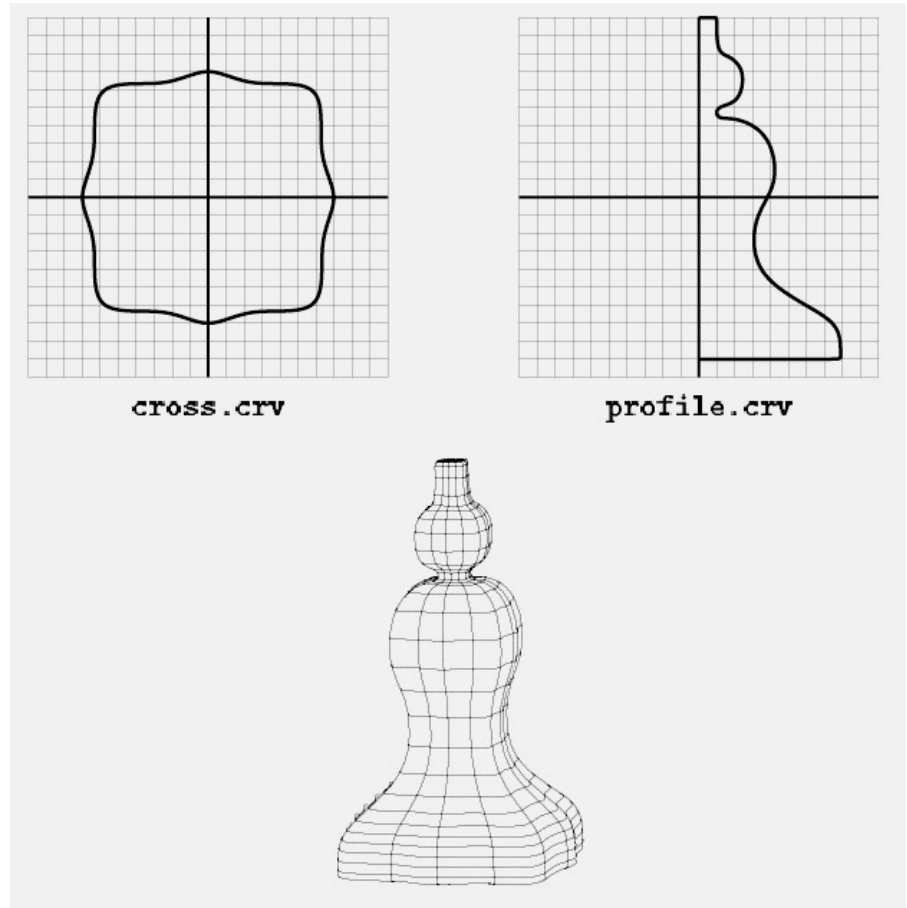
[Ching-Kuang Shene]

Generalized cylinders

- General swept surfaces
 - varying radius
 - varying cross-section
 - curved axis



[Zhou et al. SIGGRAPH Asia 2015]



[Snyder 1992]

Generalized cylinders

- spine curve $\mathbf{p}(t)$
 - normal vector $\mathbf{n}(t)$
 - binormal vector $\mathbf{b}(t)$
- cross-section curve $\mathbf{c}(t, s) = (c_1(t, s), c_2(t, s))$
- generalized cylinder surface:

$$\mathbf{f}(s, t) = \mathbf{p}(t) + c_1(t, s)\mathbf{n}(t) + c_2(t, s)\mathbf{b}(t)$$