Monte Carlo Illumination

CS 4620 Lecture 20

Surface illumination integral (as sum)

- **BRDF tells you how light from a single direction is reflected**
- **Light coming from a small source behaves similarly**
- **What about light coming from everywhere?**
  - approximate incoming light with many small sources on a sphere (the little bug can't tell the difference…)
  - reflected light is sum of reflected light due to each source (each source has its size Ω_k, brightness L_k, and direction ω_k)

$$L_r(\omega_r) = \sum_k \Omega_k L_k f_r(\omega_k, \omega_r) |\omega_k \cdot \mathbf{n}|$$

- $L_r(\omega_r)$: reflected light in direction ω_r
- $\Omega_k L_k$: "intensity" of light source k
- $f_r(\omega_k, \omega_r)$: BRDF
- $|\omega_k \cdot \mathbf{n}|$: cosine factor

Surface illumination integral

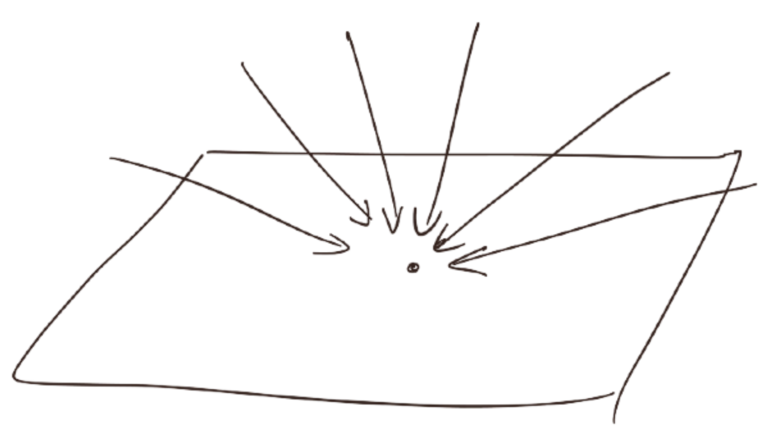
- **Take the limit as the little area sources get smaller**
  - collection of separate brightnesses L_k becomes a function $L_i(\omega_i)$
  - size of sources turns into an integration measure $d\sigma$

$$L_r(\omega_r) = \int_{S^2_+} L_i(\omega_i) f_r(\omega_i, \omega_r) |\omega_i \cdot \mathbf{n}| d\sigma(\omega_i)$$

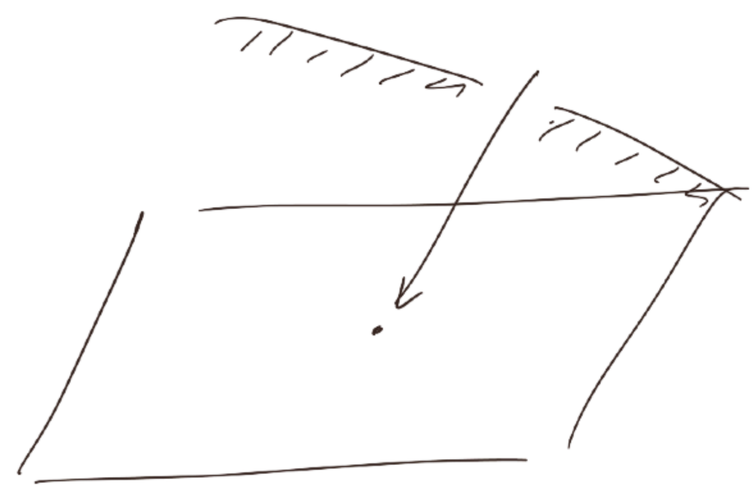
"The light reflected to direction ω_r is the integral, over the positive unit hemisphere, of the incoming light times the BRDF times the incoming cosine factor, with respect to surface area."

A word on radiometric units

- **Power**
  - energy per unit time, Watts
- **Irradiance**
  - energy per unit area, W/m^2
- **Radiance**
  - energy per unit area and per unit solid angle, $W/(m^2\ sr)$



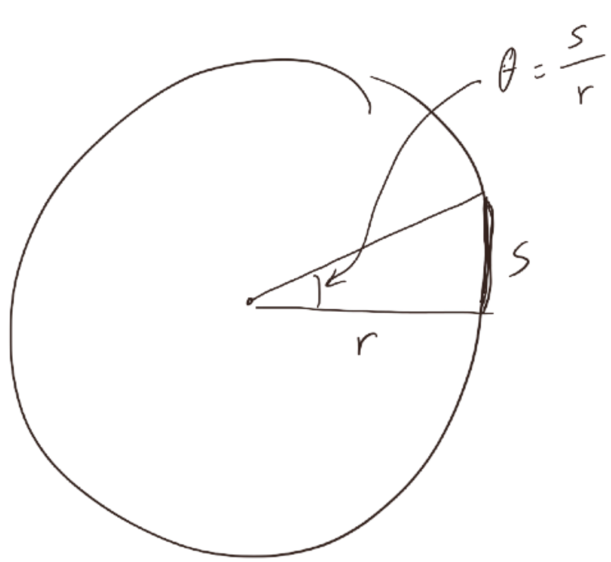
irradiance



radiance

Angle and solid angle

- **Angle**
  - size of a set of 2D directions (subset of unit circle)
  - length / distance; whole circle has angle 2π radians
- **Solid angle**
  - the size of a set of 3D directions (subset of unit sphere)
  - area / distance^2; whole sphere has solid angle 4π *steradians*



Monte Carlo Integration

- **Monte Carlo idea: design a random experiment whose average outcome is the answer we want**
- **Integration:**

$$I = \int_a^b f(x)dx$$

- **want to define an "estimator"** *g*(*x*) **such that**

$$E\{g(x)\} = I$$ for random values of *x*

  - **that is, the expected value of g is the answer we seek when** ***x*** **is chosen randomly.**

Uniform sampling

- **If *x* is chosen uniformly at random from [*a*, *b*]:**

$$E\{f(x)\} = \frac{1}{b-a}\int_a^b f(x)dx$$

- so, to get the desired answer, set

$$g(x) = (b-a)f(x)$$

- then

$$E\{g(x)\} = \int_a^b f(x)dx = I \qquad \text{for } x \text{ uniform in } [a, b]$$

Aside: probability density functions

- **Probability distribution: familiar notion in the discrete case**
  - a distribution divides up one unit of probability among the elements of a *probability space*.
  - e.g. roll two dice; probability space is $\Omega = \{1, \ldots, 6\}^2$
  - each possible roll is equally likely: $p((i,j)) = \frac{1}{36}$
  - probability distribution p has to be normalized: $\sum_{x \in \Omega} p(x) = 1$
  - a random variable is a function on Ω
  - e.g. sum of the two dice: $S((i,j)) = i + j$
  - values of S are distributed over $\{2, \ldots, 12\}$
  - $S \sim p_S \quad \text{where} \quad p_S(n) = \Pr\{S(x) = n\}$

Aside: probability density functions

- **Probability distribution can also be over a continuous set**
  - e.g. spin a spinner from 0 to 6; probability space is $\Omega = [0, 6)$
  - each possible spin is equally likely: $p(x_0) = \frac{1}{6} = \frac{\Pr\{x_0 < x < x_0 + dx\}}{dx}$
  - probability density p has to be normalized: $\int_\Omega p(x) dx = 1$
  - a random variable is a function on Ω
  - e.g. sum of two spins: $S : \Omega^2 \to \mathbb{R} : S(x, y) = x + y$
  - values of S are distributed over $[0, 12)$
  - $S \sim p_S \quad \text{where} \quad p_S(z)\, dz = \Pr\{z < S(x, y) < z + dz\}$

$$p(0) = 0; \quad p(1) = \frac{1}{36}; \quad p(6) = \frac{1}{6}; \quad p(12) = 0$$

Expectation

- **Discrete case**

$$\text{when } x \sim p(x),\ E\{f(x)\} = \sum_{x \in \Omega} f(x)p(x)$$

- **Continuous case**

$$\text{when } x \sim p(x),\ E\{f(x)\} = \int_{\Omega} f(x)p(x)\,dx$$

Uniform sampling revisited

- **Choosing points uniformly from [a, b] is sampling from a pdf that has density 1 / (b – a).**
  – if we use an estimator g with uniformly sampled x:

$$E\{g(x)\} = \int_a^b g(x)p(x)\,dx = \frac{1}{b-a}\int_a^b g(x)dx$$

  – so if f is the desired integrand, the correct estimator is

$$g(x) = (b-a)f(x)$$

Convergence rate

- **We can get a better estimate of the expected value of *g* by generating several values and averaging them.**

$$G_n = \frac{1}{N} \sum_{i=1}^{n} g(x_i) \quad \text{where } x_i \sim p$$

- **As *n* increases, the variance of G_n decreases**

$$\sigma^2 \left\{ \sum_{i=1}^{n} g(x_i) \right\} = \sum_{i=1}^{n} \sigma^2\{g\} = N\sigma^2\{g\}$$

$$\sigma\{G_n\} = \frac{\sigma\{g\}}{\sqrt{N}}$$

Nonuniform sampling

- **Choosing points instead from some other distribution over the interval [a, b] also works just as well**
  - if we use an estimator g with $x \sim p(x)$

$$E\{g(x)\} = \int_a^b g(x)p(x)\,dx$$

  - so if f is the desired integrand, the correct estimator is

$$g(x) = \frac{f(x)}{p(x)}$$

$$E\{g(x)\} = \int_a^b \frac{f(x)}{p(x)}p(x)\,dx = \int_a^b f(x)dx$$

as long as p(x) is not zero!

Monte Carlo illumination

- **Monte Carlo integration is widely used to compute illumination integrals**
  - integrand: product of illumination and BRDF and cosine factor

$$L_r(\omega_r) = \int_{S^2_+} L_i(\omega_i) f_r(\omega_i, \omega_r) |\omega_i \cdot \mathbf{n}| d\sigma(\omega_i)$$

  - if we choose:

$$\omega_i \sim p(\omega_i) \quad \text{and set:} \quad g(\omega_i) = \frac{L_i(\omega_i) f_r(\omega_i, \omega_r) |\omega_i \cdot \mathbf{n}|}{p(\omega_i)}$$

  - then: $E\{g(\omega_i)\} = L_r(\omega_r)$ (as long as $p > 0$ over the whole hemisphere)
  - this is an algorithm for computing L_r!

Example: uniform sampling

- **If we select directions uniformly over the hemisphere…**
  - then:

    $$p(\omega_i) \sim 1/(2\pi)$$

  - 2π because that is the area (solid angle) of the hemisphere; that way, probability integrates to 1
  - the correct estimator is:

    $$g(\omega_i) = \frac{L_i(\omega_i) f_r(\omega_i, \omega_r) |\omega_i \cdot \mathbf{n}|}{p(\omega_i)}$$

    $$= 2\pi L_i(\omega_i) f_r(\omega_i, \omega_r) |\omega_i \cdot \mathbf{n}|$$

(see notebook for how…)

Example: cosine-proportional sampling

- **If we select directions proportional to** $|\omega_i \cdot \mathbf{n}|$ (see notebook for how…)
  - then:

    $$p(\omega_i) \sim |\omega_i \cdot \mathbf{n}|/\pi$$

  - factor of π needed so that probability integrates to 1
  - the correct estimator is:

$$g(\omega_i) = \frac{L_i(\omega_i) f_r(\omega_i, \omega_r) |\omega_i \cdot \mathbf{n}|}{p(\omega_i)}$$

$$= \pi L_i(\omega_i) f_r(\omega_i, \omega_r)$$

Example: BRDF-proportional sampling

- **Suppose we can choose directions proportional to** $f_r(\omega_i, \underline{\omega_o})$ **(where ω_o is fixed and ω_i is the variable)**
  - then:

$$p(\omega_i) \sim f_r(\omega_i, \omega_o)/M(\omega_o) \quad \text{where } M(\omega_o) = \int_{S^2_+} f_r(\omega_i, \omega_o) d\sigma(\omega_i)$$

  - normalization factor needed so that probability integrates to 1
  - the correct estimator is:

$$\begin{aligned} g(\omega_i) &= \frac{L_i(\omega_i) f_r(\omega_i, \omega_r) |\omega_i \cdot \mathbf{n}|}{p(\omega_i)} \\ &= M(\omega_o) L_i(\omega_i) |\omega_i \cdot \mathbf{n}| \end{aligned}$$

