CS4620/5620: Lecture 29

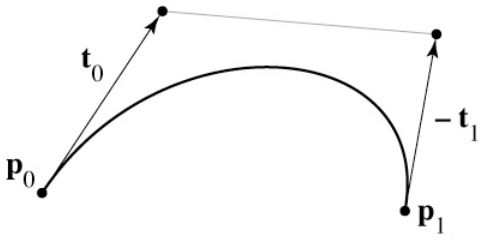
Splines

Announcements

- 4621
  - Today
  - Next Friday
- PPA 2 will be released this weekend
- HW 3 will be released on Monday

Hermite to Bézier

$$\mathbf{p}_0 = \mathbf{q}_0$$
$$\mathbf{p}_1 = \mathbf{q}_3$$
$$\mathbf{v}_0 = 3(\mathbf{q}_1 - \mathbf{q}_0)$$
$$\mathbf{v}_1 = 3(\mathbf{q}_3 - \mathbf{q}_2)$$



$$\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\\ \mathbf{d}\end{bmatrix} = \begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix} \begin{bmatrix}\mathbf{q}_0\\ \mathbf{q}_1\\ \mathbf{q}_2\\ \mathbf{q}_3\end{bmatrix}$$

Bézier matrix

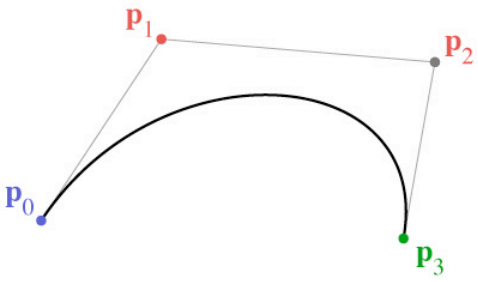
$$\mathbf{p}(t) = \begin{bmatrix}t^3 & t^2 & t & 1\end{bmatrix} \begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix} \begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}$$

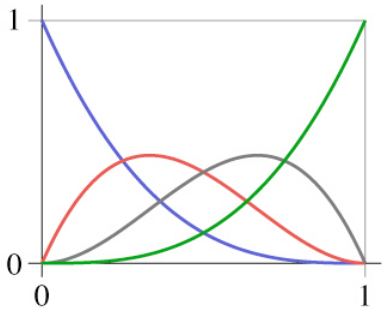
– note that these are the Bernstein polynomials

$$C(n,k)\, t^k\, (1-t)^{n-k}$$

and that defines Bézier curves for any degree

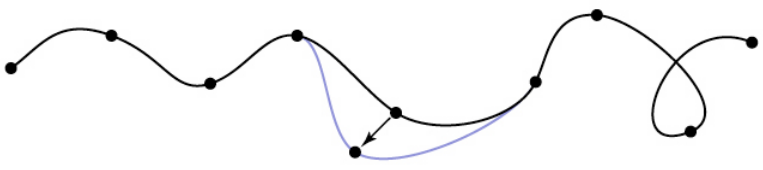
Bézier basis





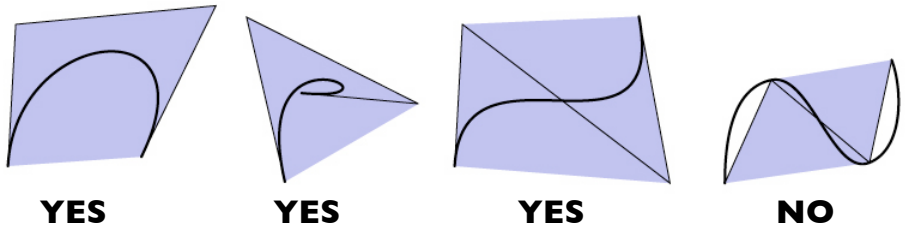
Control

- Local control
  - changing control point only affects a limited part of spline
  - without this, splines are very difficult to use
  - many likely formulations lack this



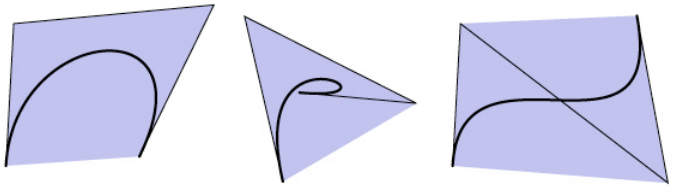
Control

- Convex hull property
  - convex hull = smallest convex region containing points
    - think of a rubber band around some pins
  - some splines stay inside convex hull of control points
    - simplifies clipping, culling, picking, etc.



Convex hull

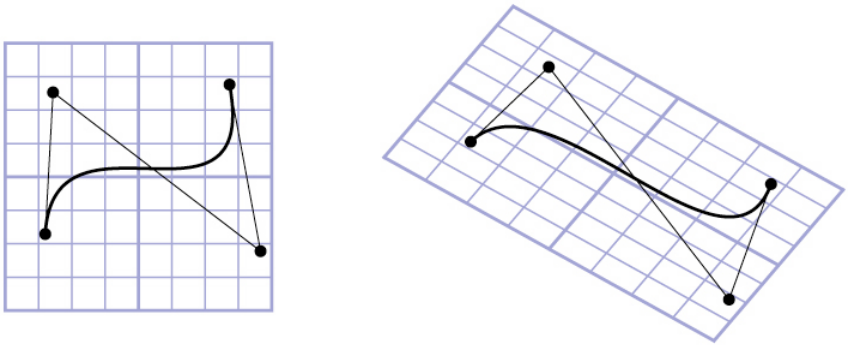
- If basis functions are all positive and sum to 1, the spline has the convex hull property



  - if any basis function is ever negative, no convex hull prop.
    - proof: take the other three points at the same place

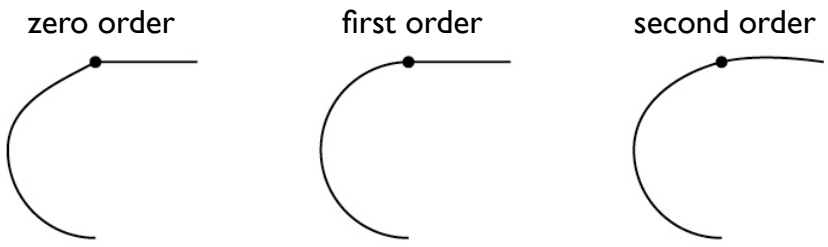
Affine invariance

- Transforming the control points is the same as transforming the curve
  - extremely convenient in practice…



Continuity

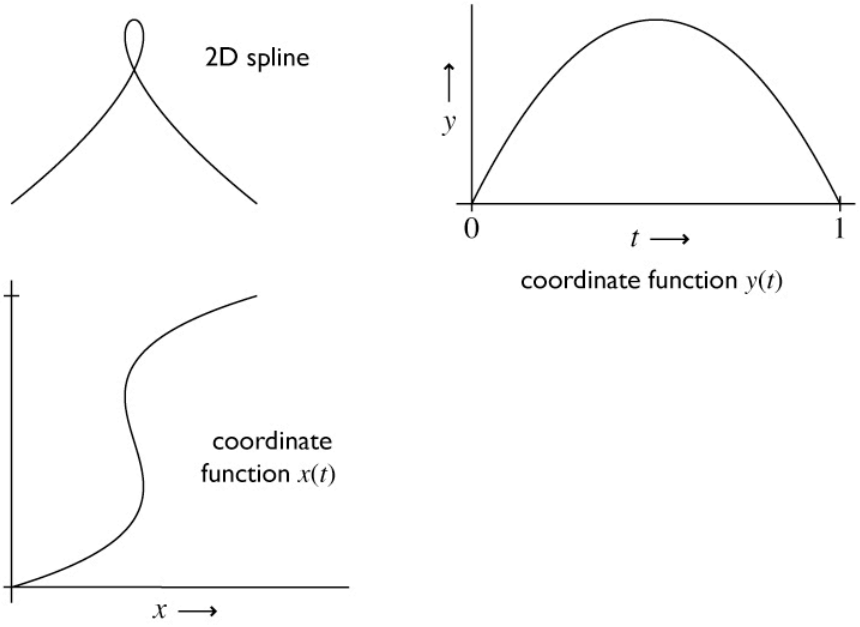
- Smoothness can be described by degree of continuity
  - zero-order (G^0): position matches from both sides
  - first-order (G^1): tangent also matches from both sides
  - second-order (G^2): curvature also matches from both sides
  - G^n vs. C^n



Continuity

- Parametric continuity (C)
  - is continuity of coordinate functions, e.g., $x(t)$, $y(t)$, $z(t)$
- Geometric continuity (G)
  - is continuity of the geometric curve itself
- Neither form of continuity is guaranteed by the other
  - Typically C^1 implies G^1
    - Can be C^1 but not G^1 when $\mathbf{p}(t)$ comes to a halt (next slide)
  - Can be G^1 but not C^1 when the tangent vector changes length abruptly

Geometric vs. parametric continuity



Continuity

$$\mathbf{p}^{(n)}(t) = \frac{d^n \mathbf{p}(t)}{dt^n}$$

- A curve is said to be C^n continuous if **p**(t) is continuous, and all derivatives of **p**(t) up to and including degree n have the same direction and magnitude:

$$\lim_{x \to t_-} \mathbf{p}^{(m)}(x) = \lim_{x \to t_+} \mathbf{p}^{(m)}(x), \quad m = 0 \ldots n$$

- G^n continuity is like C^n but only requires the derivatives to have the same direction:

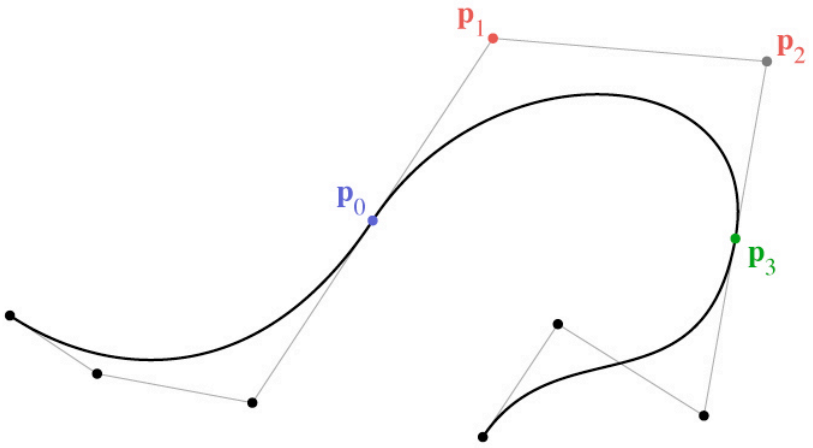
$$\lim_{x \to t_-} \mathbf{p}^{(n)}(x) = k \lim_{x \to t_+} \mathbf{p}^{(n)}(x), \quad \text{for some } k > 0$$

Chaining spline segments

- Hermite curves are convenient because they can be made long easily
- Bézier curves are convenient because their controls are all points and they have nice properties
  - but they interpolate every 4th point, which is a little odd

Chaining Bézier splines

- No continuity built in
- Achieve C^1 using collinear control points



Cubic Bézier splines

- Very widely used type, especially in 2D
  - e.g. it is a primitive in PostScript/PDF
- Can represent C^1 and/or G^1 curves with corners
- Can easily add points at any position

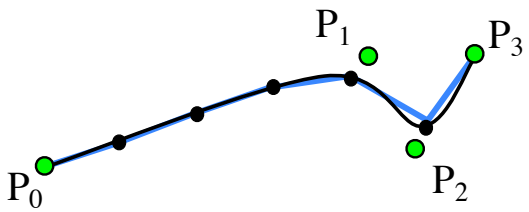
- Disadvantage
  - Special points
  - Only C^1

Evaluating splines for display

- Need to generate a list of line segments to draw
  - generate efficiently
  - use as few as possible
  - guarantee approximation accuracy
- Approaches
  - recursive subdivision (easy to do adaptively)
  - uniform sampling (easy to do efficiently)

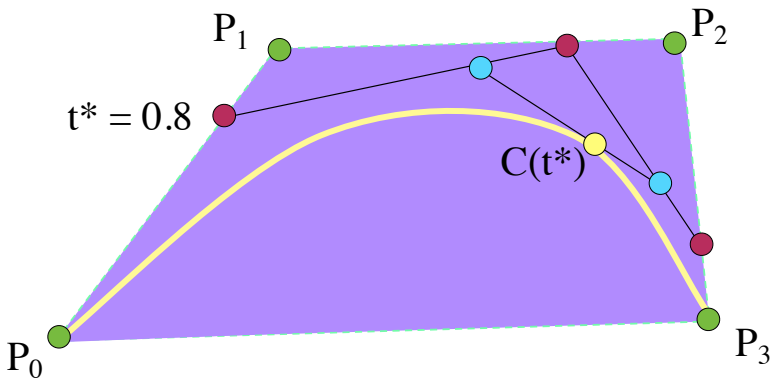
Rendering the curve: Option 1

- Uniformly sample in t
- Problem
  - may oversample smooth regions: slow
  - may undersample highly curved regions: faceted rendering



Interpolation property

- C(t*) can be evaluated using interpolation
- $C(t) = (1-t)^3 P_0 + 3 t (1-t)^2 P_1 + 3 t^2 (1-t) P_2 + t^3 P_3$

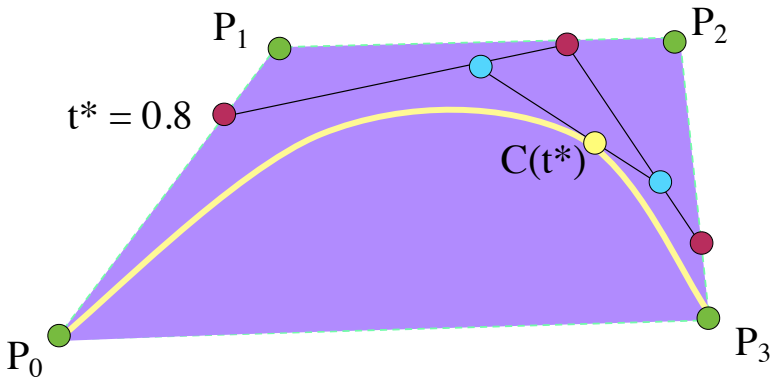


Interpolation property

$RedPt = (1-t) P_i + t P_{i+1}$

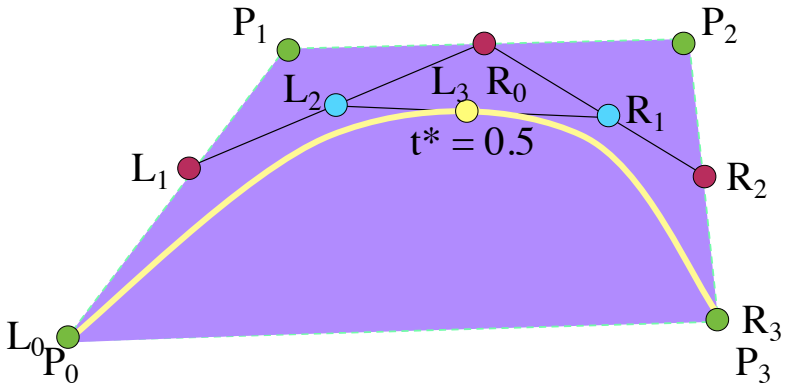
$BluePt = (1-t) RedPt_i + t RedPt_{i+1}$

$YellowPt = (1-t) BluePt_i + t BluePt_{i+1}$



De Casteljau algorithm

- Adaptive subdivision!

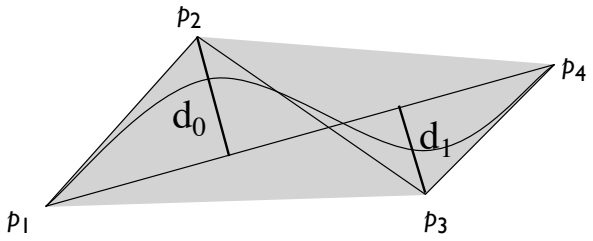


Recursive algorithm: Option 2

```
void DrawRecBezier (float eps) {
  if Linear (curve, eps)
    DrawLine (curve);
  else
    SubdivideCurve (curve, leftC, rightC);
    DrawRecBezier (leftC, eps);
    DrawRecBezier (rightC, eps);
}
```

Termination Criteria

- Test for linearity
  - distance between control points
  - distance of control points from line



$d_0 < \varepsilon$
$d_1 < \varepsilon$

B-splines

- We may want more continuity than C^1
- B-splines are a clean, flexible way of making long splines with arbitrary order of continuity
- Various ways to think of construction
  - a simple one is convolution
- An approximating spline

Deriving the B-Spline

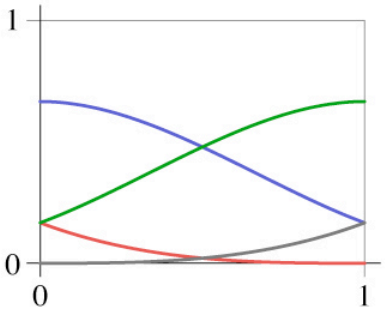
- Approached from a different tack than Hermite-style constraints
  - Want a cubic spline; therefore 4 active control points
  - Want C^2 continuity
  - Turns out that is enough to determine everything

Cubic B-spline matrix

$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{k-1} \\ \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{p}_{k+2} \end{bmatrix}$$

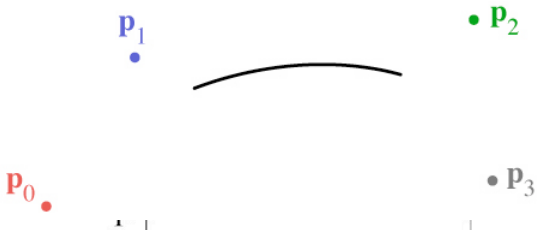
Cubic B-spline curves

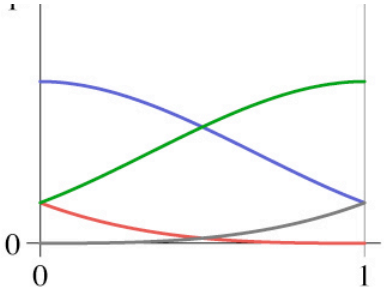
- Treat points uniformly
- C^2 continuity
- $C(t) = [(1-t)^3 P_{i-3} + (3t^3 - 6t^2 + 4)P_{i-2} + (-3t^3 + 3t^2 + 3t + 1) P_{i-1} + t^3 P_i]/6$
- Notice blending functions still add to 1



Cubic B-spline basis

- B-spline from each 4-point sequence matches previous, next sequence with C^2 continuity!
- Treats all points uniformly





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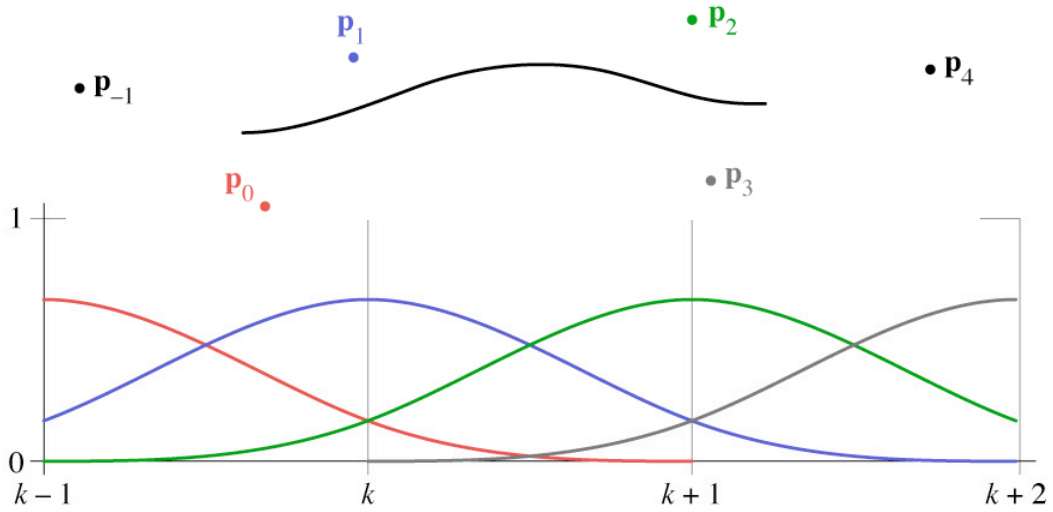
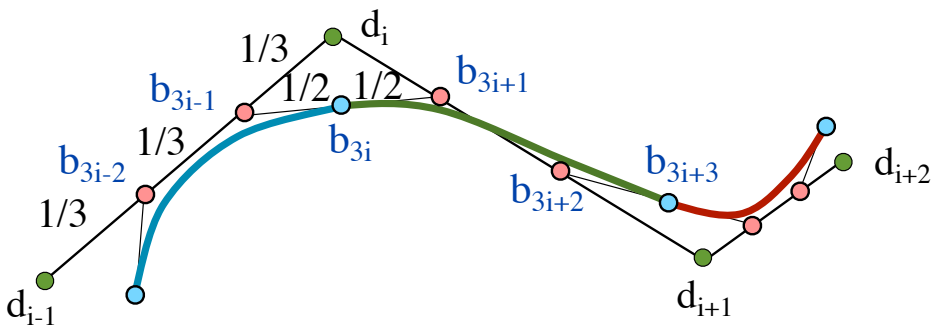


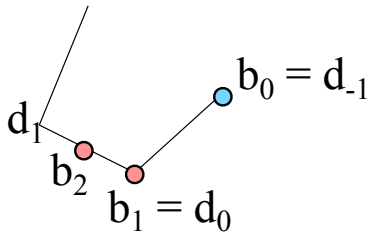
Rendering the spline-curve

- Given B-spline points $d_{-1}, \ldots d_{L+1}$
- Compute Bézier points $b_0, \ldots b_{3L}$
- Use De Casteljau algorithm to render



Equations and boundary conditions

- Equations
- $b_{3i} = (b_{3i-1}+b_{3i+1})/2$
- $b_{3i-1} = d_{i-1}/3 + 2d_i/3$
- $b_{3i-2} = 2d_{i-1}/3 + d_i/3$



- Boundary conditions
  - $b_0 = d_{-1}, b_1 = d_0, b_2 = (d_0+d_1)/2$
  - $b_{3L} = d_{L+1}, b_{3L-1} = d_L, b_{3L-2} = (d_{L-1}+d_L)/2$