CS 4620 Homework 4: Transformations

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Problem 1: Polar Decomposition (An exercise in finding closest rotations)

The *polar decomposition* of a square, real-valued matrix A, is

$$A = R\,U \tag{1}$$

where R is an orthogonal *rotation matrix* (with $\det(R) = +1$), and U is a symmetric, positive semi-definite matrix affectionately known as the *right stretch matrix*. Intuitively, it says that you can interpret any linear transformation A as first stretching vectors by U, then applying a unique rotation R[1]. The polar decomposition always exists, and is unique; it provides a way to extract the best rotational approximation to a linear transformation. In this question, you will compute the polar decomposition for a transformation you estimate.

1. Consider the transformation of four homogeneous points from

$$\mathbf{p}_0 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad \mathbf{p}_1 = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \quad \mathbf{p}_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \tag{2}$$

to

$$\mathbf{p}'_0 = \begin{pmatrix}0.1\\0\\0\\1\end{pmatrix}, \quad \mathbf{p}'_1 = \begin{pmatrix}0.1\\1.1\\0\\1\end{pmatrix}, \quad \mathbf{p}'_2 = \begin{pmatrix}-1.1\\0\\0.1\\1\end{pmatrix}, \quad \mathbf{p}'_3 = \begin{pmatrix}0.1\\0.1\\1\\1\end{pmatrix}. \tag{3}$$

Estimate the affine transformation,

$$\mathbf{T} = \begin{bmatrix}A & v\\0 & 1\end{bmatrix}, \tag{4}$$

such that $\mathbf{p}'_i = \mathbf{T}\mathbf{p}_i, \forall i$. Setup the implied linear system, and solve it for $\mathbf{T}$. You can solve the system numerically—document your work.

2. Compute the polar decomposition of the 3-by-3 block matrix, A, using the following five steps (document your work):

   (a) First, construct the symmetric, positive semidefinite matrix, $U^2 = A^T A$, thereby eliminating R by exploiting the fact that $R^T R = I$.

   (b) Construct the eigenvalue decomposition (c.f. §5.4),

$$U^2 = \sum_{i=1}^{3} \lambda_i \hat{v}_i \hat{v}_i^T, \tag{5}$$

   where $\lambda_i \geq 0$ are eigenvalues, and $\hat{v}_i$ are orthonormal vectors.

[1] There is also a second polar decomposition, $A = W R$, that uses a left-stretch matrix, W, which stretches *after* rotating. Interestingly, the rotation matrix, R, is the same in each case.

(c) Construct the right symmetric stretch tensor, U, using a matrix square root of U^2, which is easy to evaluate given the eigenvalue decomposition,

$$U = \sum_{i=1}^{3} \lambda_i^{\frac{1}{2}} \hat{v}_i \hat{v}_i^T. \tag{6}$$

(d) Construct the inverse of U using the eigenvalue decomposition,

$$U^{-1} = \sum_{i=1}^{3} \lambda_i^{-\frac{1}{2}} \hat{v}_i \hat{v}_i^T \tag{7}$$

(e) Finally, construct the rotation matrix by matrix multiplication,

$$R = AU^{-1}. \tag{8}$$

Roughly speaking, what does this particular rotation do? *(Tip: You might want to verify that your rotation matrix has* $\det(R) = 1$*)*