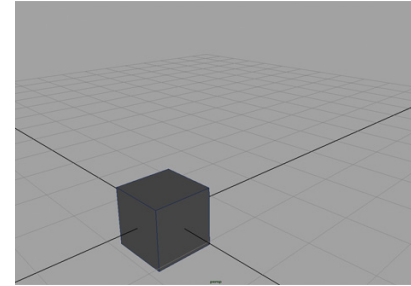


3D Transformations

CS 4620 Lecture 7

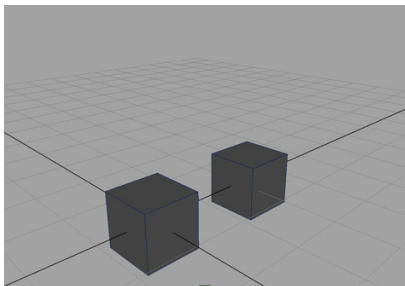
Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



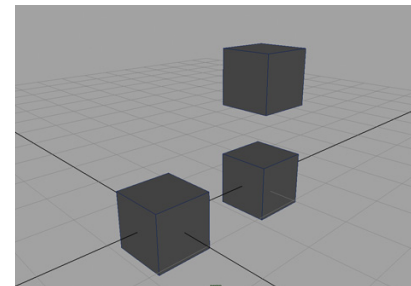
Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



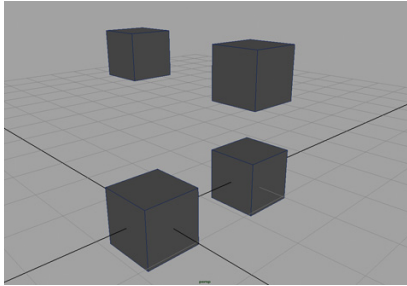
Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



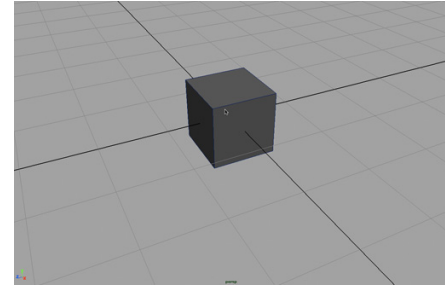
Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



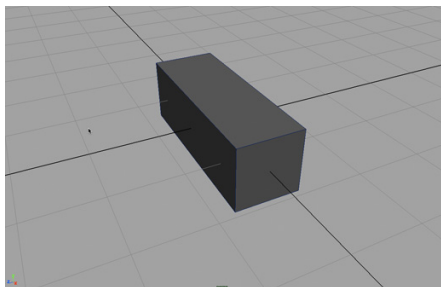
Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



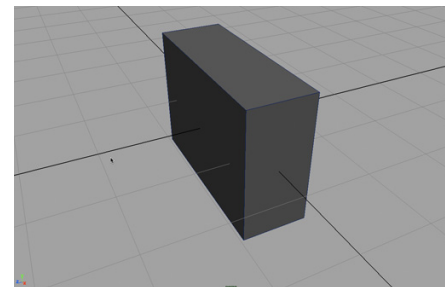
Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



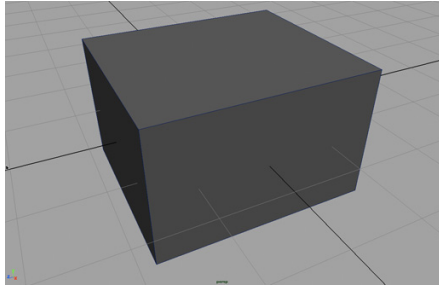
Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



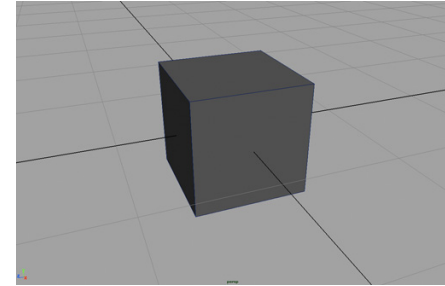
Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



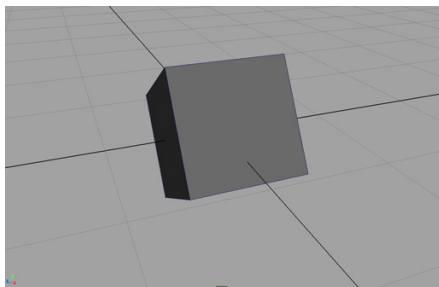
Rotation about z axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



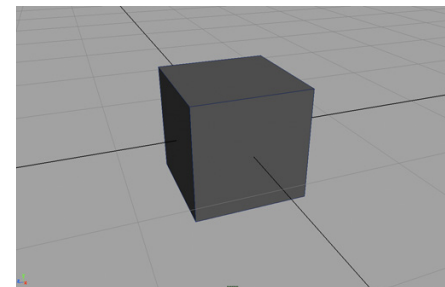
Rotation about z axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



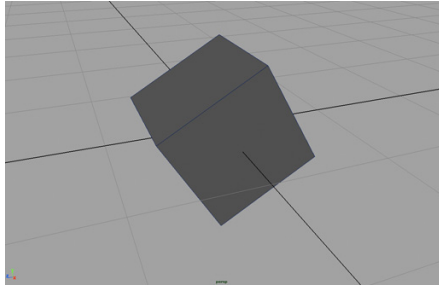
Rotation about x axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



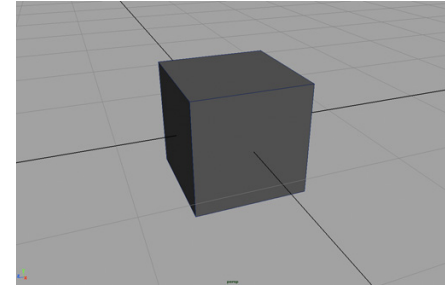
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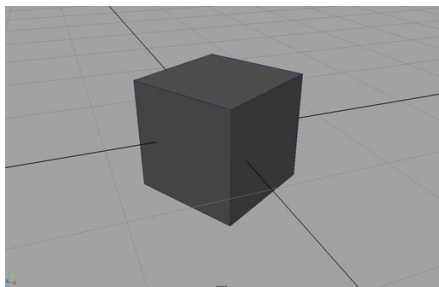
Rotation about y axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



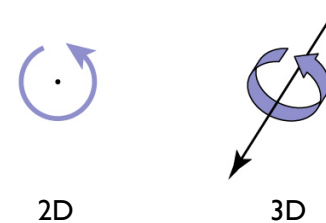
Rotation about z axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



General rotations

- A rotation in 2D is around a point
- A rotation in 3D is around an axis
 - so 3D rotation is w.r.t a line, not just a point
 - there are many more 3D rotations than 2D
 - a 3D space around a given point, not just 1D



Specifying rotations

- In 2D, a rotation just has an angle
 - if it's about a particular center, it's a point and angle
- In 3D, specifying a rotation is more complex
 - basic rotation about origin: unit vector (axis) and angle
 - convention: positive rotation is CCW when vector is pointing at you
 - about different center: point (center), unit vector, and angle
 - this is redundant: think of a second point on the same axis...
- Alternative: Euler angles
 - stack up three coord axis rotations

Coming up with the matrix

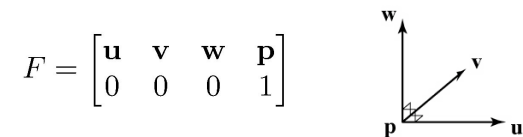
- Showed matrices for coordinate axis rotations
 - but what if we want rotation about some random axis?
- Compute by composing elementary transforms
 - transform rotation axis to align with x axis
 - apply rotation
 - inverse transform back into position
- Just as in 2D this can be interpreted as a similarity transform

Building general rotations

- Using elementary transforms you need three
 - translate axis to pass through origin
 - rotate about y to get into x-y plane
 - rotate about z to align with x axis
- Alternative: construct frame and change coordinates
 - choose p, u, v, w to be orthonormal frame with p and u matching the rotation axis
 - apply similarity transform $T = F R_x(\theta) F^{-1}$

Orthonormal frames in 3D

- Useful tools for constructing transformations
- Recall rigid motions
 - affine transforms with pure rotation
 - columns (and rows) form right handed ONB
 - that is, an **orthonormal basis**



$$F = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Building 3D frames

- Given a vector **a** and a secondary vector **b**
 - The **u** axis should be parallel to **a**; the **u-v** plane should contain **b**
 - $\mathbf{u} = \mathbf{a} / \|\mathbf{a}\|$
 - $\mathbf{w} = \mathbf{u} \times \mathbf{b}$; $\mathbf{w} = \mathbf{w} / \|\mathbf{w}\|$
 - $\mathbf{v} = \mathbf{w} \times \mathbf{u}$
- Given just a vector **a**
 - The **u** axis should be parallel to **a**; don't care about orientation about that axis
 - Same process but choose arbitrary **b** first
 - Good choice is not near **a**: e.g. set smallest entry to 1

Building general rotations

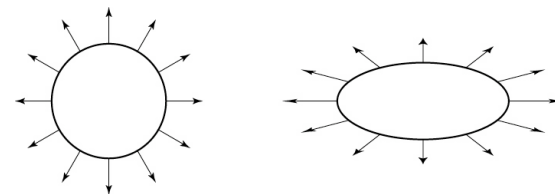
- Alternative: construct frame and change coordinates
 - choose p, u, v, w to be orthonormal frame with p and u matching the rotation axis
 - apply similarity transform $T = F R_x(\theta) F^{-1}$
 - interpretation: move to x axis, rotate, move back
 - interpretation: rewrite u -axis rotation in new coordinates
 - (each is equally valid)

Building transforms from points

- Recall 2D affine transformation has 6 degrees of freedom (DOFs)
 - this is the number of “knobs” we have to set to define one
- Therefore 6 constraints suffice to define the transformation
 - handy kind of constraint: point **p** maps to point **q** (2 constraints at once)
 - three point constraints add up to constrain all 6 DOFs (i.e. can map any triangle to any other triangle)
- 3D affine transformation has 12 degrees of freedom
 - count them by looking at the matrix entries we're allowed to change
- Therefore 12 constraints suffice to define the transformation
 - in 3D, this is 4 point constraints (i.e. can map any tetrahedron to any other tetrahedron)

Transforming normal vectors

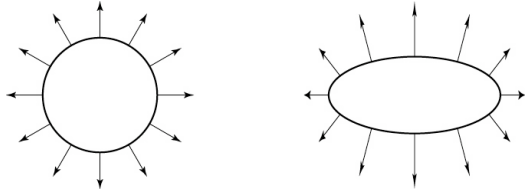
- Transforming surface normals
 - differences of points (and therefore tangents) transform OK
 - normals do not



have: $\mathbf{t} \cdot \mathbf{n} = \mathbf{t}^T \mathbf{n} = 0$
 want: $M\mathbf{t} \cdot X\mathbf{n} = \mathbf{t}^T M^T X\mathbf{n} = 0$
 so set $X = (M^T)^{-1}$
 then: $M\mathbf{t} \cdot X\mathbf{n} = \mathbf{t}^T M^T (M^T)^{-1} \mathbf{n} = \mathbf{t}^T \mathbf{n} = 0$

Transforming normal vectors

- Transforming surface normals
 - differences of points (and therefore tangents) transform OK
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