CS 4620 Homework 3 Solutions

1. A math joke (from http://haha.nu/funny/funny-math/, where you can find more jokes):

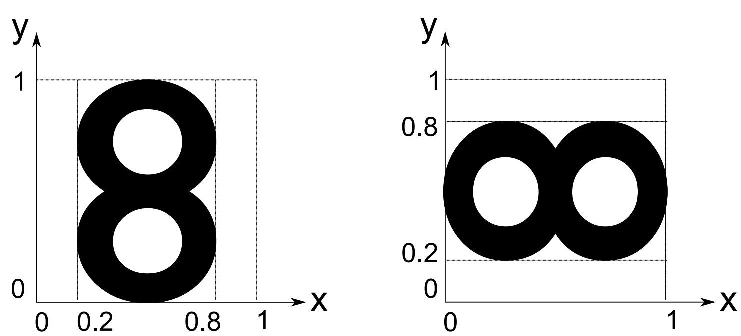
> After explaining to a student through various lessons and examples that:
>
> $$\lim_{x \to 8} \frac{1}{x-8} = \infty$$
>
> I tried to check if the student really understood that, so I gave a different example. This was the result:
>
> $$\lim_{x \to 5} \frac{1}{x-5} = \rotatebox{90}{5}$$

From the view today the student actually learnt to perform a 2D transformation. Let's explore the transformation in the joke. In the following questions, you are required to express all the tranformations by 3x3 homogeneous transformation matrices.

(a) Suppose the characters "8" and "∞" are both symmetric in the particular font (Fig. 1), there are four different transformations that can transform from figure 1(a) to 1(b). Compute these transformations.

(b) Among the four transformations, only one can transform from figure 2(a) to 2(b). Point out which one.

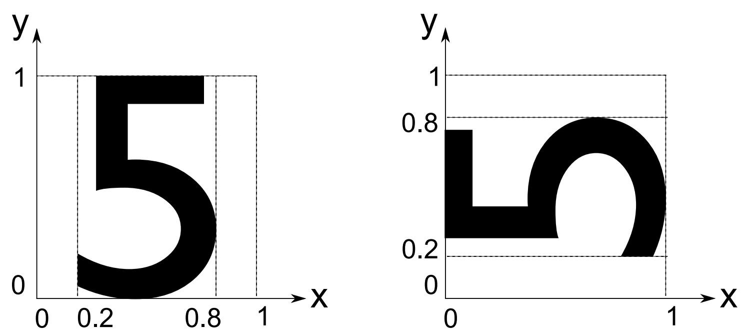
(c) Choosing different origins will change the transformation matrix. Compute the transformation matrix for figure 2, if we place the new origin at the point $(0.8, 0)$.



(a) (b)

Figure 1: Characters "8" and "∞"

Answer: (a) The four transformations are (1) Translate by $(-0.5, -0.5)$, rotate $90°$ counterclockwise, and then translate back by $(0.5, 0.5)$; (2) Translate by $(-0.5, -0.5)$, rotate $90°$



(a) (b)

Figure 2: Characters “5” and “$\rotatebox{90}{5}$”

clockwise, and then translate back by $(0.5, 0.5)$; (3) Translate by $(-0.5, -0.5)$, reflect against y axis, rotate $90°$ counterclockwise, and then translate back by $(0.5, 0.5)$; (4) Translate by $(-0.5, -0.5)$, reflect against y axis, rotate $90°$ clockwise, and then translate back by $(0.5, 0.5)$. Write them in matrices, we have

Translate $(-0.5, -0.5)$

$$T_1 = \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

Translate $(0.5, 0.5)$

$$T_2 = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotate $90°$ counterclockwise

$$R_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotate $90°$ clockwise

$$R_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflect against y axis

$$S = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally we have the four transformations are

$$M_1 = T_2 R_1 T_1 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_2 = T_2 R_2 T_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_3 = T_2 R_1 S T_1 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_4 = T_2 R_2 S T_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) To transform from figure 2(a) to figure 2(b), we need rotate 90° counterclockwise, so it is the first one, M_1.

(c) Under the new coordinates, we need first translate $(0.8, 0)$ to make the character in the same position as shown in figure 2(a), we then do transformation M_1, and get the same image as shown in figure 2(b), finally we translate it back by $(-0.8, 0)$ to the new axis. Thus the transformation matrix is

$$M_1' = \begin{bmatrix} 1 & 0 & -0.8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M_1 \begin{bmatrix} 1 & 0 & 0.8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0.2 \\ 1 & 0 & 0.8 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Explore the commutativity of transformations. For two 3D transformations $\mathbf{A}$ and $\mathbf{B}$, we say $\mathbf{A}$ and $\mathbf{B}$ are commutative if their compositions $\mathbf{AB} = \mathbf{BA}$. Now define the following three 3D transformations:

   i. Rotation $\mathbf{R}$ which transforms $+x$ axis to $+y$, $+y$ to $+z$, and $+z$ to $+x$;

   ii. Scale $\mathbf{S}$ which scales along x, y and z axes by factors 2, 3 and 4, respectively;

   iii. Translation $\mathbf{T}$ which moves along x, y and z axes by 3, 2 and 1 units, repsectively.

   Questions:

   (a) Write $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{T}$ in 4x4 homogeneous matrices;

   (b) Compute the compositions $\mathbf{RS}$, $\mathbf{SR}$, $\mathbf{ST}$, $\mathbf{TS}$, $\mathbf{TR}$ and $\mathbf{RT}$. Is $\mathbf{R}$ and $\mathbf{S}$ commutative? How about $\mathbf{S}$ and $\mathbf{T}$, $\mathbf{T}$ and $\mathbf{R}$?

   (c) Fill in the following table with commutativity of transformations in general. For instance, if any rotation is commutative with any translation, fill row 1 col 3 with $\surd$, otherwise $\times$.

	Rotation	Scale	Translation
Rotation			
Scale			
Translation			

(d)* For all $\times$ cells in the table, find out the sufficient and necessary conditions under which the corresponding types of transformations becomes commutative.

Answer: (a) We first write the rotation in 3x3 matrix $\mathbf{R}'$. Note after $\mathbf{R}'$ the vector $(1,0,0)^{\mathrm{T}}$ becomes $(0,1,0)^{\mathrm{T}}$, $(0,1,0)^{\mathrm{T}}$ becomes $(0,0,1)^{\mathrm{T}}$, and $(0,0,1)^{\mathrm{T}}$ becomes $(1,0,0)^{\mathrm{T}}$. Thus

$$\mathbf{R}' \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{R}' \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{R}' \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Put them together we have

$$\mathbf{R}' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

thus

$$\mathbf{R}' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Write it as a 4x4 homogeneous matrix, we have

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As for $\mathbf{T}$ and $\mathbf{S}$, they are straightforward

$$\mathbf{S} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) In this problem we simply do matrices multiplications

$$\mathbf{RS} = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{SR} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{ST} = \begin{bmatrix} 2 & 0 & 0 & 6 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{TS} = \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{TR} = \begin{bmatrix} 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{RT} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) From the result above, for rotation and translation, translation and scale, and scale and rotation, none of them is commutative. It is also easy to see two translations, and two scales are both commutative. Now the question is if two rotations are commutative. The answer is no. Consider another rotation

$$\mathbf{R}' = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is easy to verify $\mathbf{R}'$ and $\mathbf{R}$ are not commutative. So finally the table is

	Rotation	Scale	Translation
Rotation	$\times$	$\times$	$\times$
Scale		$\surd$	$\times$
Translation			$\surd$

(d) First of all, an identical translation is always commutative with the other transformation. Thus, in the following discussion, we only consider the case in which none of the two transformations is identical.

For translation $\vec{x} \to \vec{x} + \vec{t}$ and any linear transformatin $\vec{x} \to \mathbf{P}\vec{x}$, they are commutative means $\mathbf{P}(\vec{x}+\vec{t}) = \mathbf{P}\vec{x}+\vec{t}$, thus $\mathbf{P}\vec{t} = \vec{t}$. Consequently, they are commutative if and only if $\vec{t}$ is a fixed point under the transformation $\mathbf{P}$.

Now consider translation and rotation. $\vec{t}$ is a fixed point under the rotation if and only if $\vec{t}$ is on the axis of the rotation. As a result, we say translation and rotation is commutative if and only if the translation vector and rotation axis is collinear.

As for the translation and scale, $\vec{t} = (t_1, t_2, t_3)^{\mathrm{T}}$ is a fixed point under the scale (s_1, s_2, s_3) if and only if $s_i t_i = t_i$, i.e. $(s_i - 1)t_i = 0$ for $i = 1, 2, 3$. In a descriptive language, we say they are commutative if and only if for each axis, either the translation doesn't move along the axis, or scale doesn't change on the axis.

For arbitrary scale $\mathbf{S}'$ and rotation $\mathbf{R}'$, consider a point $\vec{x}$ on the rotation axis. Note $\vec{x}$ is a fixed point under the rotation. So $\mathbf{R}'\mathbf{S}' = \mathbf{S}'\mathbf{R}'$ leads to $\mathbf{R}'\mathbf{S}'\vec{x} = \mathbf{S}'\mathbf{R}'\vec{x}$, which is $\mathbf{R}'(\mathbf{S}'\vec{x}) = \mathbf{S}'\vec{x}$. Note $\mathbf{R}'$ is not identical, so $\mathbf{S}'\vec{x}$ is also on the axis of $\mathbf{R}'$. We can interpret this necessary condition in descriptive language as, if the rotation axis is not parallel to any coordinate axis, the scale must be a uniform scale; if the ration is along some coordinate axis, then the scale must be uniform on the other two axes, or it is a 180° rotation. We could verify this condition is also a sufficient condition.

For two rotations $\mathbf{R}'$ and $\mathbf{R}''$, similarly let $\vec{x}$ be a point on the rotation axis of $\mathbf{R}'$. We could similarly have $\mathbf{R}'(\mathbf{R}''\vec{x}) = \mathbf{R}''\vec{x}$, which means $\mathbf{R}''\vec{x}$ is also on the axis of $\mathbf{R}'$. Since the rotation doesn't change the length of vectors, we have either $\mathbf{R}''\vec{x} = \vec{x}$ or $\mathbf{R}''\vec{x} = -\vec{x}$. In the first case, it measn they have the same rotation axis; for the second case, it means their rotation axis are perpendicular to each other, and the rotation angles are both 180°. In descriptive language, the necessary condition is they have the same rotation axis, or the axis are perpendicular to each other and their rotation angle are both 180°. It is easy to verify the condition is also sufficient.