CS 4620 Homework 2 Solution

This homework is on perspective and ray tracing. Note 1(c) and 2(c) are starred problems.

1. Consider a scene in which a unit cube frame is placed on a horizontal plane (Fig. 1, AB $= 1$), and you are looking from some position P. Figures 2(a) and 2(b) are two possible views when you move around. Suppose in both 2(a) and 2(b) the front and back faces are both squares, and the front faces are centered in the view. Given that AB : CD : EF $= 1 : 0.8 : 2$ in 2(a), and AB : CD : EF $= 1 : 0.6 : 1.5$ in 2(b), please compute

   (a) the distance from the eye position P to the plane, for 2(a) and 2(b) respectively;

   (b) the field of view (f.o.v.) of 2(a);

   (c)* the f.o.v. of 2(b).

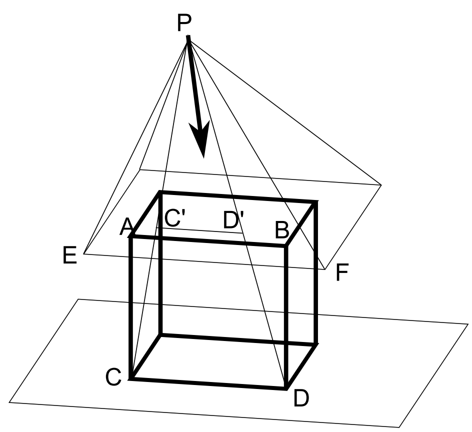
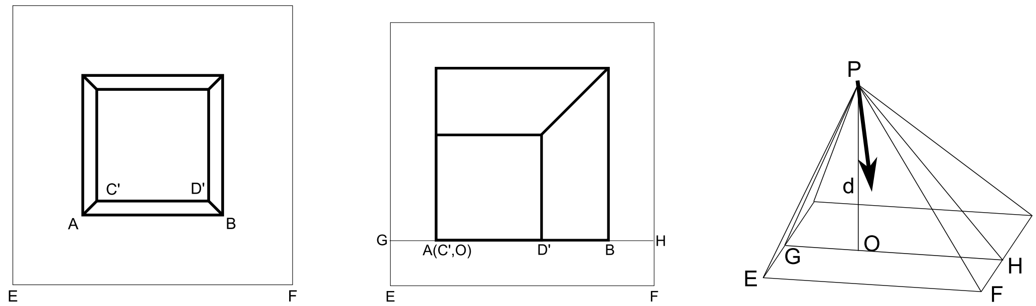


Figure 1: A unit cube frame placed on a horizontal plane.

Answer: There is a hidden trick here that in v2 it doesn't explicitly say where the projection plane is. It is totally OK if the answer simply assumes the plane is parallel to the projection plane, but we would give extra credit if it reasons why they should be. In fact, we could prove a lemma that if any parallel lines are still parallel when projected, the projection plane is parallel to these lines. We could prove this by contradiction. If the projection plane is not parallel to the lines, we draw an auxiliary line from P that parallel with the lines, then the three lines intersect at an infinite point. Consequently, those lines' projected image should also intersect with each other. Note the auxiliary line is not parallel to the projection plane, it has a finite intersection Q with the plane. Since the auxiliary line's projected image is only a point Q, the other two lines' projected image should intersect at Q. But this contradicts with the assumption that their projected images are still parallel lines. Using the lemma, the projection plane is parallel to the plane because the square on the plane is projected as a rectangle, which means the parallel image of the two pairs of edges are still parallel lines, thus the square is parallel to these two pairs of edges. Finally we prove the plane is parallel to the projection plane.



(a) On the projection plane we have AB $= 1$, C$'$D$' = 0.8$, and EF $= 2$.

(b) On the projection plane we have OG $= 0.25$, C$'$D$' = 0.6$, and OH $= 1.25$

(c) Drop a perpendicular from P and get O. Draw a parallel line from O and get G and H, then the f.o.v. equals $\angle$GPH $=$ $\angle$GPO $+$ $\angle$HPO.

Figure 2: The view on the projection plane and f.o.v. calculation.

Now we turn to question (1a). We simply set the front face as the projection plane. Suppose CD's projected image is C$'$D$'$. Note PCD and PC$'$D$'$ are similar triangles, we have $\frac{d'-1}{d'} = \frac{\mathrm{C'D'}}{\mathrm{CD}} = \frac{\mathrm{C'D'}}{\mathrm{AB}}$, in which d' is the distance from P to the plane, and $d'-1$ is the distance to the projection plane. For 2(a), we have C$'$D$' = 0.8$, therefore $d' = 5$; for 2(b), we have C$'$D$' = 0.6$, therefore $d' = 2.5$.

For question (1b), this is a head-on perspective view (not oblique) so the atan formula applies. If we set the front face as the projected plane, we have $d = 4$, $h = 2$, in which the h is the length of EF $=$ 2AB $= 2$. So the f.o.v. is $2\,\mathrm{atan}\,\frac{h}{2d} = 28.1°$.

For question (1c), this is an oblique perspective. As shown in figures 2(b) and 2(c), we have the f.o.v. equals $\angle$GPH $=$ $\angle$GPO $+$ $\angle$HPO $= \mathrm{atan}\,\frac{\mathrm{OG}}{d} + \mathrm{atan}\,\frac{\mathrm{OH}}{d} = \mathrm{atan}\,\frac{0.25}{1.5} + \mathrm{atan}\,\frac{1.25}{1.5} = 49.3°$.

2. We can often see the sunlight on water at the lakeside or seashore (Fig. 3). To analyze the scene, let's consider a simple model (Fig. 4), in which 4(a) is the description of the model and 4(b) is our field of view. Our model regards the water as a plane W with a lot of waves on the surface, and ignores the fact that the Earth is round. Suppose we are standing on the bank and looking from position E towards the direction V. The viewing direction $\vec{\mathrm{EV}}$ has an angle $\beta = 5°$ with the water. Also suppose our eyes have a maximum vertical view angle $\alpha = 30°$, which means A and B in 4(a) correspond to the top and bottom lines in 4(b). The sun can be regarded as an infinite far away object whose angular size is $\delta = 0.5°$, and the center of the sun is $\gamma = 5°$ above the water. In other words, we can simply think of the sunlight as a beam of light whose angle with the water ranges in $\gamma \pm \frac{\delta}{2}$.

   (a) In the field of view (Fig. 4(b)), H is the total height of the view, S is the portion of the view occupied by the sky, D_1 is the distance from the center of the sun to the top, and d is the diameter of the sun. Please compute $\frac{S}{H}$, $\frac{D_1}{H}$ and $\frac{d}{H}$.

   (b) What's the shape of the inverted image of the sun if the water surface is perfectly flat?

   (c)* Now take the waves into consideration. Suppose the waves have a maximum slope of $4°$. The inverted image will be stretched because larger areas of water surface can reflect the sunlight to us. Let L denote the length of the inverted image, and D_2 is the distance from the bottom of the sun's image to the edge of the view. Please compute $\frac{L}{H}$ and $\frac{D_2}{H}$.

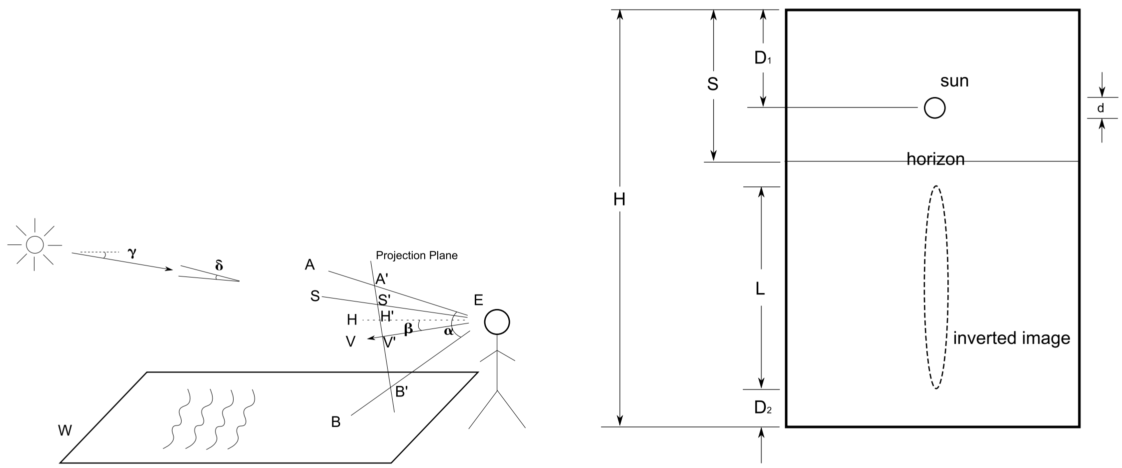
(d) A small sailboat appears in our view from far away. We find the height of the boat looks roughly the same as the diameter of the sun, but its actual height is 16ft. Please approximately compute the distance to the boat. Again, ignore the fact that the Earth is round.

Figure 3: Sun Light on water from http://www.alaska-in-pictures.com

Answer: Draw the projection plane which is perpendicular to the viewing direction, at a distance of 1 to the eye. For any point P, the vertical position of its projected image P′ on the projection plane, $\mathrm{V'P'} = \tan\angle\mathrm{PEV}$. Therefore, $\mathrm{V'H'} = \tan 5°$, $\mathrm{V'A'} = \mathrm{V'B'} = \tan 15°$, and $\mathrm{V'S'} = \tan 10°$, in which S stands for the direction of the center of the sun. Consequently we have $\frac{S}{H} = \frac{\mathrm{A'H'}}{\mathrm{A'B'}} = \frac{\mathrm{A'V'}-\mathrm{H'V'}}{\mathrm{A'B'}} = \frac{\tan 15°-\tan 5°}{2\tan 15°} = 0.3367$. Similarly $\frac{D_1}{H} = \frac{\tan 15°-\tan 10°}{2\tan 15°} = 0.1710$, $\frac{d}{H} = \frac{\tan 10.25°-\tan 9.75°}{2\tan 15°} = 0.0168$.

As for (2b), the inverted image is exactly circular. This is only true when the center of the inverted image lies on the viewing direction. For other positions of the sun, the inverted image is an ellipse.

For (2c), consider rays which come out of the eye, reflect on the water, and then hit the sun. If the water was totally flat, only the rays which fall in the angle range of $\pm 0.25°$ w.r.t. viewing direction would have a chance to hit the sun. If we take the waves into consideration, note a $4°$ change on normal vector would cause a $8°$ change on the reflection ray, we would conclude that the rays which fall into the range of $-8.25°$ to $5°$ would have a chance to hit the sun, in



(a) The model (b) Field of view

Figure 4: The model for the scene in figure 3

which $5°$ is the angle of horizon, the upper bound of the inverted image, w.r.t. the viewing direction. Thus $\frac{D_2}{H} = \frac{\tan 15° - \tan 8.25°}{2\tan 15°} = 0.2294$, and $\frac{L}{H} = \frac{\tan 5° + \tan 8.25°}{2\tan 15°} = 0.4338$.

Finally, for (2d), suppose the eye is at $h$ft above the water, and the boat is $d$ft away from us. Then the top of the boat is $\operatorname{atan}\frac{16-h}{d}$ above the horizon in angle, thus $5° + \operatorname{atan}\frac{16-h}{d}$ w.r.t. the viewing direction. Similarly, the angle of the bottom of the boat w.r.t. the viewing direction is $5° - \operatorname{atan}\frac{h}{d}$. Thus the height of the boat on the projection plane is $\tan(5° + \operatorname{atan}\frac{16-h}{d}) - \tan(5° - \operatorname{atan}\frac{h}{d}) = \tan 10.25° - \tan 9.75°$, where the right hand side is the height of the sun on the projection plane. Then it leads to a quadratic equation $d^2 + (2h-16)d - h(16-h)(\tan 5°)^2 = \frac{16(1+\tan^2 5°)}{\tan 10.25° - \tan 9.75°}d \approx 1791.8d$, which results in $d = 1793.2$ if $h = 0$, $d = 1792.1$ if $h = 6$, $d = 1791.8$ if $h = 8$ and $d = 1790.4$ if $h = 16$.

However, you don't have to do such a complicated computation. We can simply assume the eye is at 8ft, and assume the angular size of the sun ($0.5°$) is the same as the angular size of the boat, which is $2\operatorname{atan}\frac{8}{d}$, and we will get $d = 1833.5$. Actually we would allow any reasonable simplification to the question and accept the answer within 5% error.