

CS 4220: Numerical Analysis

Midterm

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The oral midterm should take about ten minutes, and will consist of two questions. The first question will be taken from the list below (chosen at random). There will then be a follow-up question which will build on the initial question. One sheet of hand-written notes (front and back) is allowed. You may also ask for hints (for partial credit). The exam overall will be ten points, with the following rubric:

- 2 points: Student communicated timing clearly (e.g. signing up on the sheet) and showed up on time.
- 4 points per problem on the following scale
 - 4 points: Gave a clear and correct answer with no prompting (though clarifying questions to me are allowed)
 - 3 points: Gave a mostly correct answer, possibly with some minimal prompting, hints, or consulting a reference
 - 2 points: Got the crux of the answer, but potentially with significant gaps or major hints required
 - 1 points: Appeared to understand the question, really didn't get the answer
 - 0 points: Blank stare or skipped question

I anticipate the class median will probably be around 7-8 of 10.

Questions:

1. Consider the function $f_n(x) = ((1+x)^n - 1)/x$ where $|x| \ll 1$ and n is large. Direct evaluation of this formula tends to lose a lot of accuracy – based on your understanding of floating point error analysis, give an error estimate that explains why.
2. Argue that if $A, E \in \mathbb{R}^{n \times n}$ with $\|E\| \leq \epsilon \|A\|$ and $\kappa(A)\epsilon < 1$, then $A + E$ is invertible.
3. Suppose $E \in \mathbb{R}^{n \times n}$ is a matrix that admits cheap matrix-vector products and that $\|E\| < 1$ in some induced norm. Describe how to (approximately) solve $(I - E)x = b$ using a Neumann series expansion, without ever forming E explicitly.

- Suppose $Q : [0, 1] \rightarrow \mathbb{R}^{n \times n}$ and $Q(s)^T Q(s) = I$ for all $s \in [0, 1]$. Argue that $Q'(s) = Q(s)S(s)$ for some skew-symmetric matrix $S(s)$.
- Suppose you are given a fast routine $\text{invA}(b)$ that solves $Ax = b$ for nonsingular $A \in \mathbb{R}^{n \times n}$. Describe how you might use this to efficiently solve a bordered system

$$\begin{bmatrix} A & b \\ c^T & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- Consider the equation

$$\begin{bmatrix} A & b \\ c^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Argue that if the large matrix is nonsingular, that means if $y = 0$ then A is singular.

- Suppose $H \in \mathbb{R}^{n \times n}$ is an upper Hessenberg matrix. Describe how you would compute a QR factorization of H in $O(n^2)$ time.
- Let $M \in \mathbb{R}^{n \times n}$ be a positive definite matrix, and define the M -inner product $\langle x, y \rangle = y^T M x$. Describe how to solve the problem of minimizing $\|Ax - b\|_M^2$ where $A \in \mathbb{R}^{n \times r}$ for $r < n$ is a matrix with full column rank.
- Consider the least squares problem of minimizing $\|Ax - b\|_2^2$ where $A \in \mathbb{R}^{m \times n}$ has full column rank. Argue that this is equivalent to solving the linear system

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

- Suppose we compute the Householder QR factorization of a tall skinny matrix A , storing the Q implicitly as a product of reflectors $I - \tau v_j v_j^T$. How would you efficiently extend the factorization to a factorization of $\begin{bmatrix} A & b \end{bmatrix}$?