

# CS 4220: Numerical Analysis

## Practice Final

David Bindel

You may have one sheet (front and back) of notes. No devices are allowed, and phones should be silenced and kept away for the duration of the exam. All eight questions are worth 6 points; for the remaining two points, please make sure to write your name and NetID on the front page and write your NetID on each page.

Problem	Score
NetID?	
1	
2	
3	
4	
5	
6	
<i>Total</i>	

## Problem 1.

Classify each statement as true or false and give a brief explanation.

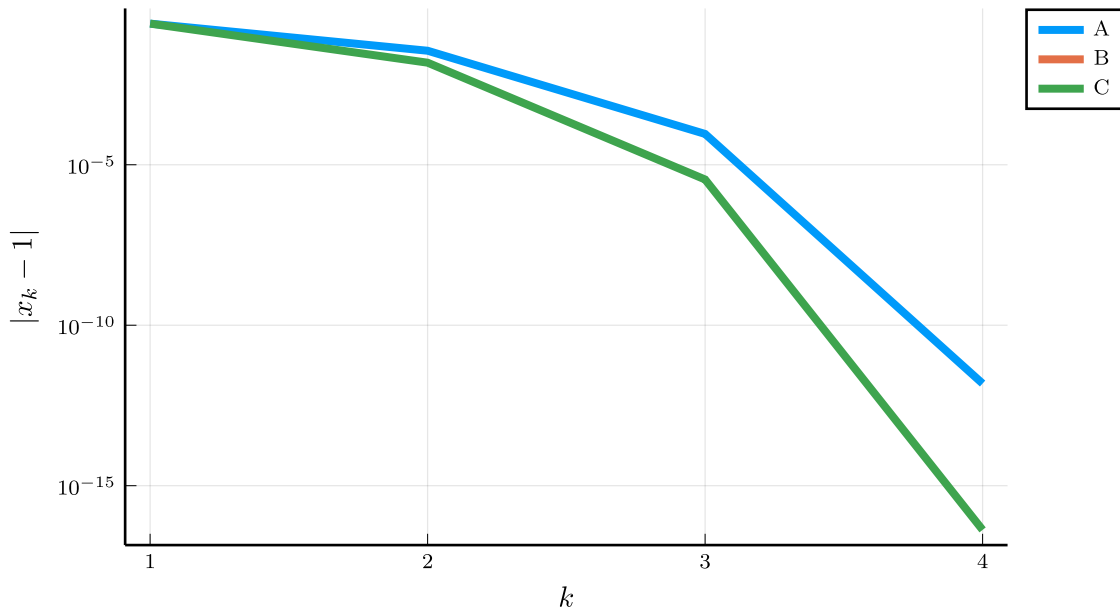
- The power method always converges to the dominant eigenvector.
- Gauss-Seidel converges in one step when the matrix  $A$  is lower triangular and nonsingular.
- An algorithm is well-conditioned if small relative changes to the inputs result in small relative changes to the outputs.
- If  $\phi$  is twice continuously differentiable and  $x^*$  is a local minimizer with a positive Hessian then Newton will converge quadratically to  $x^*$  from starting points  $x^0$  that are near enough.
- If  $A = M - N$  where  $M$  is invertible and  $\|M^{-1}N\| < 1$  for some operator norm, then  $A$  is nonsingular.
- If  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuously differentiable mapping and  $\|G(x) - G(y)\| \leq \|x - y\|$  for all  $x, y$ , then the iteration  $x^{k+1} = G(x^k)$  converges from any  $x^0$ .

## Problem 2.

The following three graphs show the convergence of three methods of minimizing  $\phi(x) = -\exp(-x^2)$ :

- Gradient descent with step size  $-0.5$ .
- Newton iteration for optimization.
- Newton iteration for finding the minimum of  $-\log(-\phi(x))$ .

All iterations start at  $x_0 = 0.25$ . Plot B is invisible because the error is exactly zero at  $x_1$ .



Explain your reasoning.

### Problem 3.

Analyze the convergence of the fixed point iteration

$$x_{k+1} = c - \exp(x_k)$$

You may use without argument that there exists a unique fixed point for any  $c$ .

- What is the equation for a fixed point?
- Under what conditions will the iteration converge with a good initial guess?
- What is the asymptotic rate of convergence?

#### Problem 4.

Consider the iteration  $Ax^{k+1} = f(x^k)$  where  $A$  is nonsingular,  $f$  has Lipschitz constant  $\alpha$  in the 2-norm, and  $\alpha < \sigma_{\min}(A)$ . Argue that the iteration converges to a unique fixed point.

## Problem 5.

Suppose  $A \in \mathbb{R}^{n \times n}$  is symmetric positive definite and  $c \in \mathbb{R}^n$  is nonzero. Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be twice continuously differentiable and  $H_g$  everywhere at least positive semidefinite. We assume the function `derivsg(z)` returns  $g(z), g'(z), g''(z)$ . We will consider optimization of

$$\phi(x) = \frac{1}{2}x^T A x + g(c^T x)$$

- Write the gradient and Hessian of  $\phi$
- Show that the minimizer satisfies  $Ax^* = cz^*$  for some  $z^* \in \mathbb{R}$ , and substitute  $x = A^{-1}cz$  into the objective to get a reduced problem in terms of  $z$
- Write a Julia loop for Newton iteration to find  $x^*$  (you may assume a good initial guess  $x^0$ ). Your code should use the reduced form, so you should only need to solve one linear system with  $A$ .

## Problem 6.

Consider minimizing the objective

$$\phi(x) = \sum_{i=1}^{n-1} c_i(x_i, x_{i+1})$$

- Argue that the Hessian is symmetric tridiagonal
- Argue that if the Hessian of each  $c_i$  is positive definite, then the Hessian of  $\phi$  is also positive definite
- Write a short Newton loop that forms the Hessian as a `SymTridiagonal` matrix in Julia (the constructor takes a vector of diagonal entries and a vector of off-diagonals). You may assume a function `derivsc(i, xi, xnext)` that computes  $c(x_i, x_{i+1})$ , its gradient, and its Hessian.

## Problem 7.

Argue that the equations

$$x = a + \alpha \cos(y)$$

$$y = b + \alpha \sin(x)$$

have a unique solution for any  $a, b \in \mathbb{R}$  and  $|\alpha| < 1$ .

## Problem 8.

Suppose  $A \in \mathbb{R}^{m \times n}$  with  $m > n$  is full column rank. Let  $A = QR$  be an full QR factorization.

- What is the minimum norm solution to  $A^T x = b$ ?
- Give an expression for all solutions to  $A^T x = b$
- Write a short Julia code to minimize  $\frac{1}{2}x^T H x - x^T d$  subject to  $A^T x = b$  using the null space method