Numerical analysis: Homework 4

Instructor: Anil Damle
Due: April 11, 2025

Policies

You may discuss the homework problems freely with other students, but please refrain from looking at their code or writeups (or sharing your own). Ultimately, you must implement your own code and write up your own solution to be turned in. Your solution, including plots and requested output from your code should be typeset and submitted via the Gradescope as a pdf file. This file must be self contained for grading. Additionally, please submit any code written for the assignment as zip file to the separate Gradescope assignment for code.

Question 1:

Implement Newton's method for root finding. For each of the following compute a root of the function and illustrate the order of convergence (i.e., q if $|e_{k+1}| = \rho |e_k|^q$, where $e_k = x^* - x_k$) and, if linear, the rate (i.e., ρ if $|e_{k+1}| = \rho |e_k|$) exhibited by the method. Discuss if you observe what you expect.

(a) $f(x) = x^2$

(b) $f(x) = \sin x + x^3$

(c) $f(x) = \sin \frac{1}{x}$ for $x \neq 0$

Question 2:

Consider the CG iteration from $x^{(0)} = 0$ described by Algorithm 1. Prove that as the iteration progresses $(r^{(k)})^T r^{(j)} = 0$ for $j < k$, i.e., the residuals are orthogonal.

Algorithm 1 CG

Input: SPD A and right hand side b.
Initialize $x^{(0)} = 0$, $r^{(0)} = b$, $p^{(0)} = r^{(0)}$, and $k = 1$.
while not converged **do**
  $\alpha_k = \frac{(r^{(k-1)})^T r^{(k-1)}}{(p^{(k-1)})^T A p^{(k-1)})}$
  $x^{(k)} = x^{(k-1)} + \alpha_k p^{(k-1)}$
  $r^{(k)} = r^{(k-1)} - \alpha_k A p^{(k-1)}$
  $\beta_k = \frac{(r^{(k)})^T r^{(k)}}{(r^{(k-1)})^T r^{(k-1)}}$
  $p^{(k)} = r^{(k)} + \beta_k p^{(k-1)}$
  $k = k + 1$
end while
return $x^{(k)}$

Question 3:

Prove that if $\nabla f(x) = 0$ but the Hessian $\nabla^2 f(x)$ is indefinite (i.e., has positive and negative eigenvalues), then there is a direction we can move from x that decreases the function value provided we take a small enough step. (I.e., show that x is not a local minimizer and that we can make progress when running an optimization scheme using so-called directions of negative curvature.)

Question 4 (an interesting ungraded problem):

Suppose we have a function $f : \mathbb{R}^n \to \mathbb{R}$ with a local minimizer x^* such that for any direction $p \in \mathbb{R}^n$ (say with $\|p\|_2 = 1$) there exists an $\epsilon > 0$ such that $f(x^* + \alpha p) > f(x^*)$ for all $\alpha \in (-\epsilon, \epsilon)$. Does this guarantee that x^* is a strict local minimizer of $f(x)$? Why or why not?