April 14, 2021

$$Ax = b$$

$$A = M - N \qquad Mx_{k+1} = Nx_k + b \qquad x_{k+1} = \underbrace{M^{-1}N}_{R}x_k + M^{-1}b$$

$$e_{k+1} = Re_k = R^k e_0 \qquad (e_k = x_k - x_*)$$

$$e_{k+1} \approx \rho(R)e_k \qquad \rho(R) = \max_j |\lambda_j(R)| \quad (<1 \text{ for convergence})$$

- $x_k - x_{k+1} = e_k - e_{k+1} \approx (1-\rho(R))e_k$

$$x_* = x_k - e_k \approx x_k - \frac{(x_k - x_{k-1})}{1-\rho(R)}$$

Can we do better by combining iterates?

If $e_k \to 0$, can't really do worse $\qquad x_1, \ldots, x_k \xrightarrow{Rx_k + M^{-1}b} x_{k+1}$

Recall heavy ball: $x_{k+1} = x_k - \alpha_k \nabla f_k + \beta_k(x_k - x_{k-1})$

$K_k = \text{span}\{x_0, \ldots, x_k\}$
(subspace)

$x_{k+1} = \underset{x}{\text{argmin}}\, f(x) = \frac{1}{2}x^T A x - x^T b$

s.t. $\quad x \in K_k$

1. How good is this? *
2. How to solve?
3. Efficiency? (growing k)

$x_{k+1} = \sum_{j=0} \left(\quad \right)$

$x_{k+1} = \sum_{j=0}^{k+1} c_j R^j M^{-1} b$

$K_k(A, b) = \text{span}\{b, Ab, \ldots, A^{k-1} b\}$

$K_{k+2}($

$x_{k+1} = R x_k + M^{-1} b$

$x_0 = M^{-1} b \Rightarrow x_1 = R x_0 + M^{-1} b = R M^{-1} b + M^{-1} b = (R + I) M^{-1} b$

$x_2 = R x_1 + M^{-1} b$

$= (R^2 + R + I) M^{-1} b$

$[\qquad]$

$(\qquad)e_h \quad ((\qquad))e_h$