April 14, 2021 $\quad Ax = b$

$$A = M - N \qquad Mx_{k+1} = Nx_k + b \qquad x_{k+1} = \overbrace{M^{-1}N}^{R}x_k + M^{-1}b$$

$$e_{k+1} = Re_k = R^k e_0 \qquad (e_k = x_k - x_*)$$

$$e_{k+1} \approx \rho(R)\, e_k \qquad \rho(R) = \max_j |\lambda_j(R)| \quad (< 1 \text{ for convergence})$$

- $$x_k - x_{k+1} = e_k - e_{k+1} \approx (1 - \rho(R))\, e_k$$

$$x_* = x_k - e_k \approx x_k - \frac{(x_k - x_{k-1})}{1 - \rho(R)}$$

Can we do better by combining iterates?

If $e_k \to 0$, can't really do worse $\qquad x_1, \ldots, x_k \xrightarrow{Rx_k + M^{-1}b} x_{k+1}$

Recall heavy ball: $x_{k+1} = x_k - \alpha_k \nabla f_k + \beta_k (x_k - x_{k-1})$

$K_k = \text{span}\{x_0, \dots, x_k\}$
(subspace)

$$x_{k+1} = \underset{x}{\text{argmin}}\ f(x) = \frac{1}{2}x^T A x - x^T b$$
$$\text{s.t.} \quad x \in K_k$$

1. How good is this? ☆
2. How to solve?
3. Efficiency? (growing k)

$$x_{k+1} = \sum_{j=0}^{k+1} R^j M^{-1} b \quad (R^0 = I)$$

$$x_{k+1} = \sum_{j=0}^{k+1} c_j R^j M^{-1} b$$
$$= q(R) M^{-1} b$$

$$x_{k+1} \in \text{span}\{M^{-1}b, RM^{-1}b, \dots, R^{k+1}M^{-1}b\}$$

$$\boxed{\begin{array}{l}\underline{\text{Krylov subspace}} \\ K_k(A, b) = \text{span}\{b, Ab, \dots, A^{k-1}b\}\end{array}}$$

$K_{k+2}(R, M^{-1}b)$

$$x_{k+1} = Rx_k + M^{-1}b$$

$$x_0 = M^{-1}b \Rightarrow x_1 = Rx_0 + M^{-1}b = RM^{-1}b + M^{-1}b = (R+I)M^{-1}b$$

$$x_2 = Rx_1 + M^{-1}b = R(R+I)M^{-1}b + M^{-1}b$$
$$= (R^2 + R + I)M^{-1}b$$

How good is this?

$$x_k = \underset{x}{\operatorname{argmin}} \ \tfrac{1}{2} x^T A x - x^T b \qquad (A^{-1}b)$$

$$R = V \Lambda V^{-1} \qquad p(R) = c_0 I + c_1 R + c_2 R^2 + \dots$$

$$\text{s.t.} \quad x \in K_k(R, M^{-1}b) = \operatorname{span}\{M^{-1}b, RM^{-1}b, \dots, R^{k-1}M^{-1}b\}$$

$$e_k = x_k - x_* = \underbrace{q(R)}_{\in P_{k-1}} M^{-1}b - x_* \qquad \left(x_k = \sum_{j=0}^{k-1} c_j R^j M^{-1} b \right)$$

$$= p(R) e_0 \qquad p(0) = 1$$

$$= V p(\Lambda) V^{-1} e_0 \qquad \Rightarrow M^{-1}b - x_* = e_0$$

$$\|e_k\|_2 = \|V p(\Lambda) V^{-1} e_0\| \le \kappa_2(V) \|e_0\|_2 \|p(\Lambda)\|_2$$

$$\min_{\substack{p \in P_{k-1} \\ \text{s.t.}}} \|p(\Lambda)\|_2 \} = \max_j |p(\lambda_j)| \qquad \min_p \max_j |p(\lambda_j)|$$

$$p(0) = 1$$

Chebyshev $T_0(z) = 1$

$T_1(z) = z$

$T_{n+1}(z) = 2z T_n(z) - T_{n-1}(z)$

$$\left(\frac{\sqrt{\kappa_2(A)} - 1}{\sqrt{\kappa_2(A)} + 1} \right)^k \|e_0\|$$

How to solve?

$M = I \quad N = I - A \Rightarrow x_0 = b, \quad R = I - A$

$\text{span}\{b, (I-A)b, \ldots, (I-A)^{k-1}b\} = \text{span}\{b, Ab, \ldots, A^{k-1}b\}$

$F_k = [b \;\; Ab, \ldots A^{k-1}b]$ $\quad n \times k$ basis for $K_k(A, b)$

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A x - x^T b \quad \text{s.t.} \quad x = F_k c \quad \Rightarrow \quad \min_{c \in \mathbb{R}^k} \frac{1}{2} c^T F_k^T A F_k c - c^T F_k b$$

$$\underbrace{F_k^T A F_k}_{k \times k} c = F_k^T b$$

Is F_k a good basis?

$A^k b \rightarrow$ first ovec

$\rightarrow$ ill-conditioned

$F_k = Q_k R_k \Rightarrow Q_k$ as ortho basis

$$\mathrm{Range}(AQ_k) = \mathrm{Range}\left(A\,[b \cdots A^{k-1}b]\right)$$
$$\subseteq \mathrm{Range}\left([b, Ab, \ldots, A^k b]\right) = \mathrm{Range}(Q_{k+1})$$

$$Q_{k+1} = [q_1 \cdots q_{k+1}] \Rightarrow h_{k+1,k}\, q_{k+1} = \sum_{j=1}^{k} h_{j,k}\, q_j$$

$$AQ_k e_k = Q_{k+1} \bar{H}_k e_k \qquad \bar{H}_k = \begin{array}{c} \\ \\ \\ \\ k+1 \end{array}\begin{array}{c} k \\ \begin{pmatrix} \times & \times & \times & h_{1,k} \\ \times & \vdots & \vdots & \vdots \\ 0 & \times & \times & \vdots \\ \vdots & 0 & \times & h_{k,k} \\ 0 & 0 & 0 & h_{k+1,k} \end{pmatrix} \end{array}$$

$$Q_k^T A Q_k = [Q_k \; q_{k+1}] \begin{bmatrix} H_k \\ h_{k+1,k} e_k^T \end{bmatrix} = \overset{I}{\cancel{Q_k^T Q_k}} H_k + h_{k+1,k} \overset{0}{\cancel{Q_k^T q_{k+1}}} e_k^T$$

$$Q_k^T A Q_k = H_k = \left(\quad\right) \qquad A \text{ symm} \Rightarrow H_k = \left(\quad\right)$$

$$h_{k+1,k} q_{k+1} + h_{k,k} q_k + h_{k-1,k} q_{k-1} = A q_k$$