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$$\underset{n\times n}{A}\;\underset{n\times 1}{x} = \underset{n\times 1}{b}$$

~~inv(A)·b~~ $\boxed{A\backslash b}$

$$PA = LU \Rightarrow LUx = P^T b$$

$$\min_{x\in\mathbb{R}^n} \|\underset{m\times n}{A}\;\underset{n\times 1}{x} - \underset{m\times 1}{b}\|_2^2$$

$$A = QR \Rightarrow Rx = Q^T b$$

$A\backslash b$

Direct methods: get exact answer (in exact arith) after finite number of steps

$O(n^3)$ $\quad O(mn^2)$

Iterative methods: sequence of approx $x_1, x_2, \ldots$ that (hopefully) converge

Optimization for linear systems

$$f(x) = \frac{1}{2} x^T A x - x^T b \qquad \nabla f(x) = Ax - b = 0 \iff Ax = b$$

GD: $x_{k+1} = x_k - \alpha_k (Ax_k - b)$ $\quad \|e_k\| \leq \left(\frac{\kappa_2(A) - 1}{\kappa_2(A) + 1} \right)^k$

Newton: $x_{k+1} = x_k - A^{-1}(Ax_k - b) = \cancel{x_k - x_k} + A^{-1} b$

Quasi-Newton $\quad H(x) = A \approx D, \quad D_{ii} = A_{ii} = \frac{\partial^2}{\partial x_i^2} f(x)$

$$x_{k+1} = x_k - D^{-1}(Ax_k - b)$$

Coordinate descent

$$\min_\alpha f(x + \alpha e_i) \Rightarrow \frac{1}{2}(x + \alpha e_i)^T A (x + \alpha e_i) - (x + \alpha e_i)^T b$$

$$= \frac{1}{2} \alpha^2 e_i^T A e_i + \alpha e_i^T A x - \alpha e_i^T b + \frac{1}{2} x^T A x - x^T b$$

$$\frac{\partial}{\partial \alpha} = 0 \Rightarrow 0 = \alpha A_{ii} + e_i^T (Ax - b) \Rightarrow \alpha = -(a_i^T x - b_i)/A_{ii}$$

$$[x_{k+1}]_i = [x_k]_i - (a_i^T x_k - b_i)/A_{ii}$$

for $k = 1, 2, \ldots$
  for $i = 1, 2, \ldots, n$