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Sensitivity in root finding

$$g(x_*) = 0 \qquad \hat{g}(\hat{x}_*) = \varepsilon \qquad |g(\hat{x}_*) - \hat{g}(\hat{x}_*)| \leq \delta$$

Taylor exp: $g(\hat{x}_*) \approx \cancel{g(x_*)}^{0} + g'(x_*)(\hat{x}_* - x_*)$

$$g'(x_*)(\hat{x}_* - x_*) = g(\hat{x}_*) - \hat{g}(\hat{x}_*) + \varepsilon$$

$$|\hat{x}_* - x_*| \leq \frac{\delta + |\varepsilon|}{|g'(x_*)|}.$$

$$g(\hat{x}_*) \approx \cancel{g(x_*)} + \cancel{g'(x_*)(\hat{x}_* - x_*)} + \frac{1}{2} g''(x_*)(\hat{x}_* - x_*)^2$$

$$\leq \frac{\delta + |\varepsilon|}{\frac{1}{2}|g''(x_*)|}$$

$f: \mathbb{R} \to \mathbb{R}$ $\quad \min_{x \in \mathbb{R}} f(x)$

$f: \mathbb{R}^n \to \mathbb{R}$ $\quad \min_{x \in \mathbb{R}^n} f(x)$

$f(x) = \|h(x)\|_2$ (nonlinear least squares)

Directional derivatives

Given $x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$

$$\frac{\partial f}{\partial p}(x) = \lim_{s \to 0} \frac{f(x+sp) - f(x)}{s}$$

Example: $f(y, z) = yz$

$p = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad \frac{\partial f}{\partial p} = \lim_{s \to 0} \frac{(y+s)(z+s) - yz}{s} = y + z$

$p = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \frac{\partial f}{\partial p} = \lim_{s \to 0} \frac{(\;\;+s)\cdot \;\; - \;\;}{s} = y$ $\quad \frac{\partial f}{\partial y}$

$p = e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \frac{\partial f}{\partial p} =$ $\quad z$ $\quad \frac{\partial f}{\partial z}$

local min if H is symm. pos. semidef $(\lambda_i \geq 0)$

max neg. $(\lambda_i \leq 0)$

saddle $\exists\, \lambda_i > 0,\ \lambda_j < 0$