March 26, 2021

Norms: $\|A\|_2 = \max_x \|Ax\|_2 = \sigma_1(A)$ $\quad x \approx v_1$

s.t. $\|x\|_2 = 1$

Least squares: $\arg\min_x \|Ax - b\|_2^2 \; (+ \beta \|x\|_2^2)$ $\quad \hat{x} = R^{-1} Q^T b$

$\begin{pmatrix} A \\ \sqrt{\beta} I \end{pmatrix} = QR$

Eigenvalues / eigenvectors

$\min_x x^T A x$ $\quad \lambda_n \quad Ax = \lambda_n x$

s.t. $\|x\|_2 = 1$

$\arg\min_x x^T L x$ objective function

optimization variable

s.t. $\|x\|_2^2 = n$ constraints

linear const. $e^T x = 0$

unconstrained

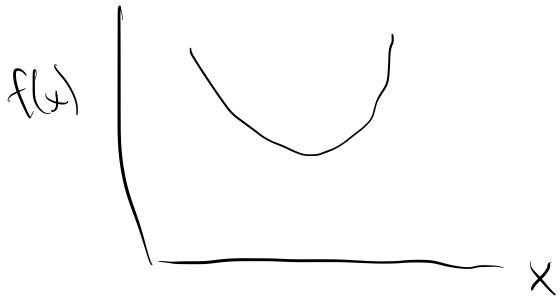
$\min_x f(x)$

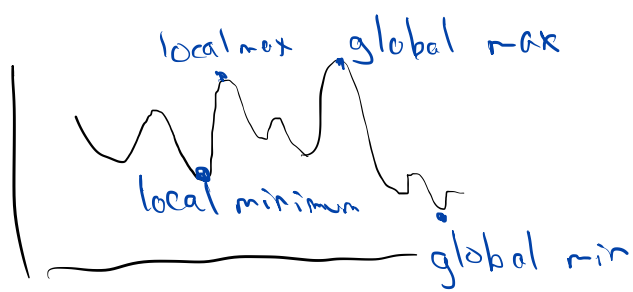
$\iff \max_x -f(x)$

constrained

$\min f(x)$

s.t. $g(x) = 0$





$f: \mathbb{R} \to \mathbb{R}$ $\quad \min_{x \in \mathbb{R}} f(x)$

Taylor expansion: $f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!} (x-a)^j$

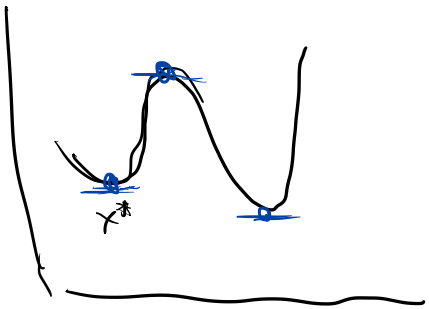
Example: e^{-x} $(a=0)$: $1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots$

$$f(x+h) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x)}{j!} h^j \approx f(x) + f'(x)h + \frac{1}{2} f''(x) h^2$$

Taylor's Theorem: $f : \mathbb{R} \to \mathbb{R}$

f' continuous, $f(x+h) = f(x) + f'(x+th)h \quad t \in (0,1)$

f'' " $f(x+h) = f(x) + f'(x)h + \frac{1}{2} f''(x+th)h^2 \quad t \in (0,1)$



Claim: x^* is local min $\Rightarrow f'(x^*) = 0$

Proof: $f'(x^*) \neq 0 \qquad p = -f'(x^*)$

$\underline{f'(x^*)p} = -[f'(x^*)]^2 < 0$

by continuity, $f'(x^* + sp)p < 0$ for $s \in [0, \varepsilon]$

by Taylor's, for $\bar{s} \in (0, \varepsilon]$ $\exists\, t \in (0,1)$

$$\underbrace{f(x^* + \bar{s}p)}_{< f(x^*)} = f(x^*) + \underbrace{f'(x^* + t\bar{s}p)\bar{s}p}_{< 0}$$

$t\bar{s} \in (0, \varepsilon)$

(same local max $p = +f'(x^*)$)

Local min or max?

f'' continuous, $f'(x^*) = 0$, $f''(x^*) > 0$

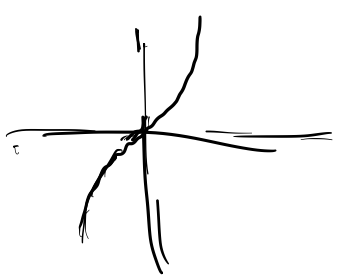
Claim: x^* is local min

Proof: By continuity, $\exists\, \varepsilon > 0$, $h \in (-\varepsilon, \varepsilon)$ $f''(x^* + h) > 0$

By Taylor's, $\underbrace{f(x^* + h)}_{> f(x^*)} = f(x^*) + \cancel{f'(x^*)}^{\,0} h + \frac{1}{2} f''(x^* + th) h^2$ > 0

$t \in (0, 1)$

$f(x) = x^3$



$f'(x) = 3x^2 = 0$

$f''(x) = 6x = 0$

$x^* = 0$

Idea: Find $f'(x^*) = 0$

check $f''(x^*) > 0$

$\Rightarrow$ local min

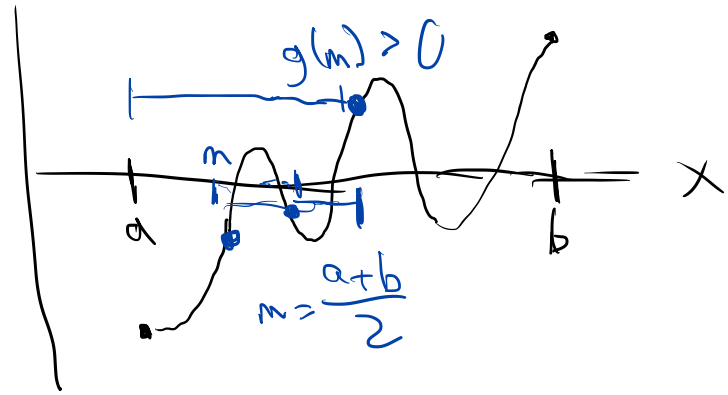
Root-finding: find x such that $g(x) = 0$ $\quad g: \mathbb{R} \to \mathbb{R}$

(Ex: $g = f'$)

Bisection

Suppose $g(a) < 0$, $g(b) > 0$

g continuous




```
bisect(a, b)
    m = (a+b)/2
    s = g(m)
    if |s| < ε  return m
    if s > 0  return bisect(a, m)
    if s < 0  return bisect(m, b)
```

after k steps, $b_k - a_k = (b_0 - a_0)/2^k$