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Norms: \( \|Ax\|_2 = \max_x \|Ax\|_2 = \sigma_1(A) \) \( x = u_1 \)

Least squares: \( \arg \min_x \| Ax - b \|^2 + \beta \| x \|^2 \) \( x = A^+ b \) \( (A^T A + \beta I)^{-1} A^T b = QR \)

Eigenvalues:
\[ \min_x x^T Ax \] \( \text{subject to} \) \( \| x \|_2 = 1 \) \( \lambda \), \( Ax = \lambda x \)

Optimization

Objective function

Unconstrained
\[ \min_x f(x) \]
\[ \Leftrightarrow \max_x -f(x) \]

Constrained
\[ \min_x f(x) \]
\[ \text{s.t.} \ g(x) = 0 \]
\[ f: \mathbb{R} \rightarrow \mathbb{R} \quad \min_{x \in \mathbb{R}} f(x) \]

Taylor expansion: \[ f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!} (x-a)^j \]

Example: \( e^{-x} \) (\( a=0 \)): \[ 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots \]

\[ f(x+h) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x)}{j!} h^j \approx f(x) + f'(x)h + \frac{1}{2} f''(x) h^2 \]
Taylor's Theorem: \( f : \mathbb{R} \to \mathbb{R} \)

\[ f(x+h) = f(x) + f'(x+h)h + \varepsilon(x) \]

\( f(x+h) = f(x) + f'(x)h + \frac{1}{2} f''(x + \theta h) h^2 + \varepsilon(x) \)

Claim: \( x^* \) is local min \( \Rightarrow f'(x^*) = 0 \)

Proof:
\[ f'(x^*) \neq 0 \quad p = -f'(x^*) \]

\[ f'(x^*) p = -[f'(x^*)]^2 < 0 \]

by continuity, \( f'(x^* + sp) p < 0 \) for \( s \in [0, \epsilon] \)

by Taylor's, for \( s \in (0, \epsilon) \) \( \exists \varepsilon \in (0, 1) \)

\[ f(x^* + sp) = f(x^*) + f'(x^* + t_s p) \overbrace{sp}^{\in (0, \epsilon)} < f(x^*) \]

\( \leq 0 \)

(same local min \( p = f'(x^*) \))
Local min or max?

\[ f^{(1)} \text{ continuous, } f'(x^*) = 0, f''(x^*) > 0 \]

Claim: \( x^* \) is local min.

Proof: by continuity, \( \exists \epsilon > 0, h \in (-\epsilon, \epsilon) \ f''(x^* + h) > 0 \)

By Taylor's, \( f(x^* + h) = f(x^*) + f'(x^*)h + \frac{1}{2} f''(x^* + \theta h)h^2 \) \( + e \in (0, 1) \)

\[ f(x) = x^3 \]

\[ f'(x) = 3x^2 = 0 \]

\[ f''(x) = 6x \]

\[ x^* = 0 \]

Root finding: find \( x \) such that \( g(x) = 0 \) \( g: \mathbb{R} \to \mathbb{R} \)

(Idea: Find \( f'(x^*) = 0 \) check \( f''(x^*) > 0 \)

\[ \Rightarrow \text{ local min} \]
Bisection

Suppose \( g(a) < 0, \ g(b) > 0 \) given \( g \) is continuous

\[
\text{bisect}(a, b) \\
m = (a + b) / 2 \\
s = g(m) \\
\text{if } |s| < \varepsilon \text{ return } m \\
\text{if } s > 0 \text{ return } \text{bisect}(a, m) \\
\text{if } s < 0 \text{ return } \text{bisect}(m, b) \\
\]

after \( k \) steps, \( b_k - a_k = (b_0 - a_0) / 2^k \)