March 19, 2021 $\quad Ax = \lambda x \quad A = S\Lambda S^{-1} \quad v_0 = Sw$

$(AS_{e_j} = S\Lambda e_j \quad As_j = \lambda_j s_j)$

Power method

Initialize $v_0 \in \mathbb{R}^n$

for $k = 0, 1, 2, \ldots$

$z_k = Av_k$

$v_{k+1} = z_k / \|z_k\|_2$

$v_k^T A v_k = \lambda v_k^T v_k$

$? \; Av_k = \lambda v_k \quad 2$

$v_{k+1} = \dfrac{\lambda v_k}{|\lambda| \|v_k\|}$

$$A^k v_0 = S\Lambda \cancel{S^{-1}} \cancel{S} \Lambda \cancel{S^{-1}} \cdots \cancel{S} \Lambda \cancel{S^{-1}} S w$$
$$= S\Lambda^k w$$
$$= \sum_{j=1}^n s_j (\lambda_j^k w_j)$$
$$= \lambda_1^k \sum_{j=1}^n s_j \left(\frac{\lambda_j}{\lambda_1}\right)^k w_j$$

$|\lambda_1| > |\lambda_2| > \ldots |\lambda_n|$

$$\|A^k v_0\|_2 = |\lambda_1|^k \left\| \sum s_j (\lambda_j/\lambda_1)^k w_j \right\|_2$$

$$z_0 = Av_0$$
$$v_1 = z_0 / \|z_0\|_2$$
$$z_1 = Av_1 = A^2 v_0 / \|z_0\|_2$$
$$v_2 = A^2 v_0 / \|z_0\|_2 \|z_1\|_2$$
$$= A^2 v_0 / \|A^2 v_0\|_2$$
$$\vdots$$
$$v_k = A^k v_0 / \|A^k v_0\|_2$$

$$v_k = \operatorname{sign}(\lambda_1)^k \frac{s_1 w_1 + \sum_{j=2}^n s_j \cancel{(\lambda_j/\lambda_1)^k} w_j}{\| \quad \|} \quad \to 0$$

Converges like $|\lambda_2/\lambda_1|^k$ to s_1

Problems?

1. sign flippage ✓
2. $w_1 = 0$?
3. $|\lambda_2/\lambda_1| = 1$
4. ≈ 1
5. only one eigenpair
6. $\mathbb{C}$?

Shift-invert

Spectral mapping theorem: f analytic, $A = S\Lambda S^{-1}$ $f(A) = S \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix} S^{-1}$

Example:

$$\left| \max_j \left| \frac{1}{\lambda_j - \sigma} \right| \right|$$