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Eigenvalue problems $A \in \mathbb{C}^{n \times n}$

$$Ax = \lambda x \quad x \in \mathbb{C}^n, \ \lambda \in \mathbb{C}$$

$$A(\alpha x) = \lambda(\alpha x)$$

$$(A - \lambda I)x = 0 \iff \det(A - \lambda I) = 0$$

Characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$$

- exactly n eigenvalues
- even if $A \in \mathbb{R}^{n \times n}$, could have $\lambda_j \in \mathbb{C} \Rightarrow \overline{\lambda_j}$ eval
- $n \geq 5 \Rightarrow$ no exact / closed form algo in ∞-precision arith

Similarity transforms

A is similar to B if $B = S^{-1} A S$ $\quad n \times n$

Claim: A, B have same evals

Proof: $\lambda x = Ax \iff \lambda \overbrace{S^{-1} x}^{y} = S^{-1} A x = \overbrace{S^{-1} A S}^{B} \overbrace{S^{-1} x}^{y}$

$$By = \lambda y \quad y = S^{-1}x \Rightarrow A(Sy) = \lambda(Sy)$$

B diagonal $\quad Be_i = B_{ii} e_i$

Jordan canonical form: $A = S J S^{-1}$

$$J = \begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_k} \end{pmatrix} \quad J_\lambda = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \quad J_\lambda e_1 = \lambda e_1$$

algebraic multiplicity: $\# \{ i \mid J_{ii} = \lambda \}$

geometric multiplicity: $\# \{ i \mid \lambda_i = \lambda \}$

$= \dim E_\lambda$

$= \dim(\{x : Ax = \lambda x\})$

Schur form

$M^H_{ij} = \overline{M_{ji}}$

Theorem: $Q^H A Q = T$, $T = \begin{pmatrix} \diagdown & * \\ & \diagdown \end{pmatrix}$ $Q^H Q = I$

$(A = Q T Q^H)$

$\det(T - \lambda I) =$

$$\begin{pmatrix} T_{11} & t_{12} & T_{13} \\ 0 & t_{ii} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix}$$

k

1

$n-k-1$

x.

$AQ = QT$

$Aq_j = q_j t_{jj}$.