

March 8, 2021

$$Q_n \dots Q_1 A = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

$$I = 2vv^T / v^T v$$

$O(mn^2)$ compute

$O(mn)$ storage

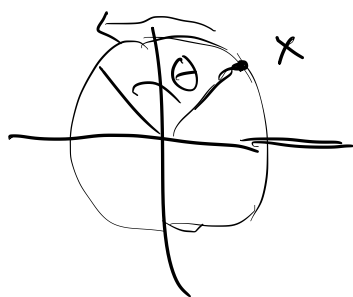
$$\hat{x} = \arg \min_x \|Ax - b\|_2 \text{ in } O(mn)$$

then QR factor $O(mn^2)$

• implicitly

★ • only multiplied by orthog. matrices

$$G = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \quad c^2 + s^2 = 1$$



$$c = \cos \theta$$

$$s = \sin \theta$$

rotate x to $\|x\|_2 e_1$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$c = x_1 / \|x\| \quad s = -x_2 / \|x\|$$

$$Gx = \frac{1}{\|x\|} \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \frac{1}{\|x\|} \begin{pmatrix} \|x\|^2 \\ 0 \end{pmatrix} = \|x\| e_1$$

Pick col $k \quad x = \begin{pmatrix} A_{ik} \\ A_{jk} \end{pmatrix}$

$$(GA)_{jk} = 0$$

$$\begin{pmatrix} x & x \\ x & x \\ x & x \end{pmatrix} \rightarrow \begin{pmatrix} x & x \\ x & x \\ 0 & x \end{pmatrix} \rightarrow \begin{pmatrix} x & x \\ 0 & x \\ 0 & x \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} x & x \\ 0 & x \\ 0 & 0 \end{pmatrix}$$

Stability

$$\hat{Q} = Q + E \quad \|E\|_2 = O(\kappa_2(Q) \cdot \varepsilon) = O(\cancel{\|Q\|_2} \cancel{\|Q^T\|_2} \varepsilon)$$

$$\begin{aligned} f(\hat{Q}A) &= f((Q+E)A) = QA + EA + F = G \\ &= Q(A + \underbrace{Q^T E A + Q^T F}_{\text{red circle}}) \end{aligned}$$

$$\|Q^T E A\| \leq \|E\| \|A\| = O(\varepsilon \|A\|)$$

$$\|Q^T F\| = \|F\| = O(\varepsilon \|A\|_2 \underbrace{\|Q+E\|_2}_{\leq \|Q\| + \|E\|}) = O(\varepsilon \|A\|)$$

Backward stable: $f(\hat{Q}A) = Q(A$

$$f(\hat{Q}_2 \hat{Q}_1 A) = f(\hat{Q}_2(Q_1 A + G_1)) \quad \|G_1\| = O(\varepsilon \|A\|)$$

$$= Q_2(Q_1 A + G_1) + G_2$$

$$= O(\varepsilon (\|Q_1\| \|A\| + \|G_1\|))$$

(least squares)

