March 5, 2021

$$\hat{x} = \underset{x}{\arg\min} \|Ax - b\|_2^2$$

$$\underset{m \times n}{\hat{A}} = \underset{m \times n}{\hat{Q}} \; \underset{n \times n}{\hat{R}} \qquad Q^T Q = I \qquad R\hat{x} = Q^T b$$

$$R = \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix}$$

Gram-Schmidt

$$A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & & \vdots \\ 0 & & \ddots & \\ & & & r_{nn} \end{pmatrix}$$

$$a_1 = q_1 r_{11} \longrightarrow q_1 = \frac{a_1}{\|a_1\|_2}, \quad r_{11} = \|a_1\|_2$$

$$a_2 = q_1 r_{12} + q_2 r_{22} \longrightarrow q_1^T a_2 = \overset{1}{\cancel{q_1^T q_1}} r_{12} + \overset{0}{\cancel{q_1^T q_2}} r_{22} \to r_{12} = q_1^T a_2$$

$$\underbrace{a_2 - q_1 q_1^T a_2}_{v_2} = q_2 r_{22}$$

$$q_2 = \frac{v_2}{\|v_2\|_2} \qquad r_{22} = \|v_2\|_2$$

$$q_1^T v_2 = q_1^T a_2 - \cancel{q_1^T q_1} q_1^T a_2 = 0$$

$$\vdots$$

$$a_j = q_1 r_{1j} + \dots + q_j r_{jj}$$

$$q_i^T a_j = q_i^T \sum_{k=1}^{j} q_k r_{kj} = r_{ij}$$

$$a_j - \sum_{k=1}^{j-1} q_k \underline{q_k^T a_j} = q_j r_{jj}$$

for $j = 1 : n$

$v_j = a_j$

for $i = 1 : (j-1)$

$r_{ij} = q_i^T a_j$

$v_j -= r_{ij} q_i$

$O(m)$

$r_{jj} = \|v_j\|_2$

$q_j = v_j / r_{jj}$

$O(mn^2)$

unstable

$r_{jj} << \|a_j\|_2$

$\hat{Q}^T \hat{Q}$ not close to I

$$A = \begin{pmatrix} x & x \\ x & x \\ x & x \end{pmatrix} \xrightarrow{Q_1 A} \begin{pmatrix} x & x \\ 0 & x \\ 0 & x \end{pmatrix} \xrightarrow{Q_2 Q_1 A} \begin{pmatrix} x & x \\ 0 & x \\ 0 & 0 \end{pmatrix}$$

$Q_i \quad m \times m$

$$Q_2 Q_1 A = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

$$A = \underbrace{Q_1^T Q_2^T}_{Q} \begin{pmatrix} R \\ 0 \end{pmatrix}$$

Efficiency in

1. multiplication by Q_i
2. $Q = Q_1^T Q_2^T$ (thin?)

$$v = -\mathrm{sign}(a_{11}) \|a_1\|_2 e_1 - a_1$$