March 1, 2021 Structured LU

$\boxed{x}$ $\boxed{x^T}$

$x^T A^T A x$
$= \|Ax\|_2^2$

Symmetric positive definite (SPD)

$A = A^T \qquad x^T A x > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$

Claim: $A = A^T$. A SPD iff all eigenvalues are positive

Proof: $A = V \Lambda V^T \qquad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$x \in \mathbb{R}^n \Rightarrow x = Vy \Rightarrow x^T A x = y^T \cancel{V^T V} \Lambda \cancel{V^T V} y = \sum_{i=1}^n y_i^2 \lambda_i$

All evals $> 0 \Rightarrow x^T A x > 0$

$\lambda_i < 0 \Rightarrow y = e_i \quad (x = v_i) \quad x^T A x = \lambda_i < 0$

Corollary: Any sub-block is SPD

Proof: $B \subseteq \{1, \dots, n\} \quad M = A[B, B] \quad x^T M x = \tilde{x}^T A \tilde{x} \qquad \tilde{x} = 0$
$\tilde{x}[B] = x$

Claim: A SPD. Then A^{-1} SPD

Proof: $A^{-1} = V \Lambda^{-1} V^T \qquad 1/\lambda_i > 0$

Cholesky factorization $\quad$ (A SPD)

$$A = L L^T \qquad L = \left(\quad \right)$$

$$\begin{pmatrix} a_{11} & a_{21}^T \\ a_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 \\ l_{21} & L_{22} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21}^T \\ 0 & L_{22}^T \end{pmatrix}$$

$$a_{11} = l_{11}^2 \Rightarrow l_{11} = \sqrt{a_{11}}$$

$$a_{21} = l_{21} l_{11} \Rightarrow l_{21} = a_{21} / \sqrt{a_{11}}$$

$$A_{22} = L_{22} L_{22}^T + l_{21} l_{21}^T$$

$$\underbrace{A_{22} - l_{21} l_{21}^T}_{S} = L_{22} L_{22}^T$$

Claim: $S^{-1} = A^{-1}[2:n, 2:n]$

$$\begin{pmatrix} a_{11} & a_{21}^T \\ a_{21} & A_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{21}^T \\ b_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix}$$

$$a_{11} b_{21}^T + a_{21}^T \qquad = 0 \qquad b_{21}^T = - a_{11}^{-1} a_{21}^T B_{22}$$

$$a_{21} b_{21}^T + A_{22} B_{22} = I$$

$$= S B$$

$A = [a_1 \cdots a_n]$

$Ax = \sum a_i x_i$

$\underbrace{\qquad}_{A_{r[i],j}} \cdot x[j]$