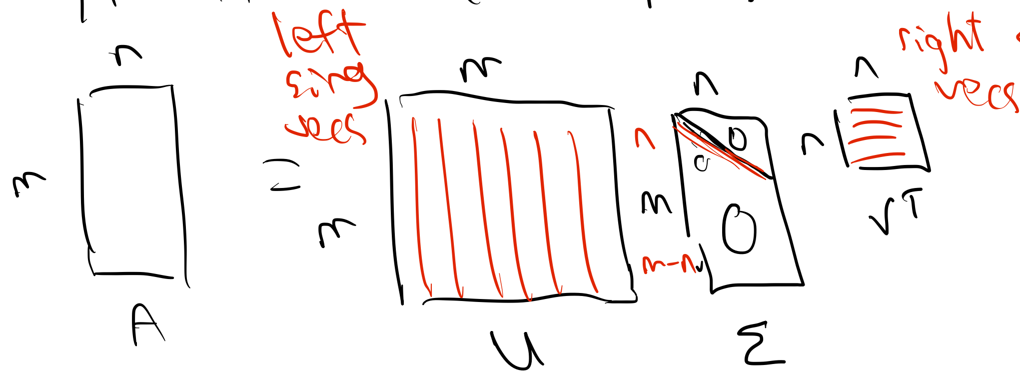
Feb 15, 2021

The Singular Value Decompositions (SVD)

$A \in \mathbb{R}^{m \times n}$ $(m \geq n)$ $A = U \Sigma V^T$

full SVD



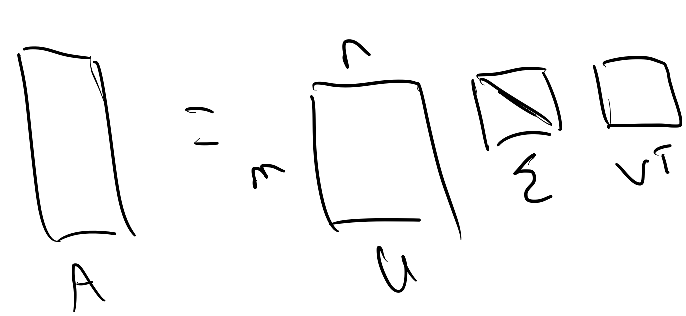
$U^T U = I = U U^T$

$V^T V = I = V V^T$

$\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$

sing. values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$

compact / reduced / thin SVD



$U^T U = I \neq U U^T$

$V^T V = I = V V^T$

$A = (U_1 \; U_2) \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^T$

$= U_1 \Sigma V^T$

outer product form: $A = \sum_{i=1}^{n} \sigma_i u_i v_i^T$

SVD and norms

$\|A\|_2 = \|U \Sigma V^T\|_2 = \|\Sigma\|_2 = \max_{\|x\|_2 = 1} \|\Sigma x\|_2 = \max \sqrt{\sum (\sigma_i x_i)^2} = \sigma_1$

$\|A\|_F = \|U \Sigma V^T\|_F = \|\Sigma\|_F = \sqrt{\sum \sigma_i^2}$ $\quad \|A\|_* = \sum_{i=1}^{n} \sigma_i$

Relationship with eigendecompositions

$$A = U \Sigma V^T$$

$$A^T A = V \Sigma U^T U \Sigma V^T = V \Sigma^2 V^T$$ eigendecomposition

$$\Sigma^2 = \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{pmatrix}$$

$$A A^T = U \Sigma^T V^T V \Sigma U^T = U \Sigma^2$$ eigendecomp

$$\Sigma^2 = \begin{pmatrix} \sigma_1^2 \end{pmatrix}$$

$$= \sqrt{\sigma_1^2} = \sigma_1 \quad \checkmark$$

$$A = A^T \Rightarrow A = X \Lambda X^T \quad X^T X = I$$

$$= X \begin{pmatrix} |\lambda_1| & & \\ & \ddots & \\ & & |\lambda_n| \end{pmatrix} \underbrace{\begin{pmatrix} \text{sign}(\lambda_1) & & \\ & \ddots & \\ & & \text{sign}(\lambda_n) \end{pmatrix} X^T}_{V^T}$$