Midterm

(due: 2020-03-13)

You may (and should) consult any resources you wish *except* for people (outside the course staff). Provide attribution for any good ideas you might get. Your final write-up should be your own.

True/false For each of the following statements, either give a brief argument that it is true or provide an example to show it is false.

1 pt The eigenvalues of a real symmetric positive definite matrix are all positive.

1 pt Every square nonsingular matrix A can be decomposed as $A = LU$ where L is unit lower triangular and U is upper triangular.

1 pt The eigenvalues of a 2-by-2 matrix are differentiable functions of the matrix entries.

1 pt For every square matrix, power iteration eventually converges to a dominant eigenvector.

1 pt If $A \in \mathbb{R}^{n \times n}$ has condition number $\kappa_2(A) = 1$, then $A = \alpha Q$ for some real $\alpha \neq 0$ and orthogonal matrix $Q \in \mathbb{R}^{n \times n}$.

1 pt If $A \in \mathbb{R}^{m \times n}$ has full column rank, then $A^\dagger A = I$.

Speedy and stable Consider each of the following:

2 pts Rewrite the following code for better speed and numerical stability assuming $PA = LU$ is already computed.

```
dx = -inv(A)*dA*inv(A)*b
```

2 pts For $x > 1$, the equation $x = \cosh(y)$ can be solved as

$$y = -\ln\left(x - \sqrt{x^2 - 1}\right).$$

What happens when $x = 10^8$? How can we fix it?

2 pts Explain the output of the following code fragment

```
x = 2;
for k = 1:100, x = sqrt(x); end
for k = 1:100, x = x^2;     end
disp(x);
```

Polynomial fitting Consider fitting a polynomial of degree at most d to n data points, i.e.

$$\text{minimize } \sum_{i=1}^{n} (p(x_i) - y_i)^2$$

2 pts Complete the following code to find $p(x) = \sum_{j=0}^{d} c_j x^j$ that solves the minimization problem by a standard linear least squares formulation:

```
function [c] = poly_least_squares(x, y, d)
```

You may write equivalent code in Julia or Python.

2 pts Modify the code to solve the problem

$$\text{minimize } \sum_{i=1}^{n} (p(x_i) - y_i)^2 + \mu \|c\|^2$$

In MATLAB, your routine should have the interface

```
function [c] = poly_least_squares_tik(x, y, d, mu)
```

You may write equivalent code in Julia or Python.

2 pts Modify the code to solve the problem

$$\text{minimize } \sum_{i=1}^{n} (p(x_i) - y_i)^2 + \mu \int_{-1}^{1} p'(x)^2 \, dx$$

In MATLAB, your routine should have the interface

```
function [c] = poly_least_squares_smooth(x, y, d, mu)
```

You may write equivalent code in Julia or Python.

Sensitive pseudoinverse Consider the problem

$$x(s) = (A + sE)^{\dagger} b$$

where $A \in \mathbb{R}^{m \times n}$ has full column rank. Suppose the economy QR factorization $A = QR$ is given.

2 pts Compute $x(0)$. Your code should take at most $O(mn)$ time. In MATLAB, this means filling in the second line in this fragment:

```
[Q,R] = qr(A,0); % This is O(mn^2)
x0 = ???;         % This should be O(mn)
```

4 pts Compute $x'(0)$ (worth two points). Your code should take at most $O(mn)$ time (worth two points). You can sanity check your code by a finite difference test:

```
h = 1e-6;
xp = (A+h*E)\b;
xm = (A-h*E)\b;
dx_fd = (xp-xm)/2/h;
dx = % your computation
relerr = norm(dx_fd-dx)/norm(dx)
```

Conditioning

2 pts Argue that if $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite and $A_{11} \in \mathbb{R}^{k \times k}$ is a leading principal submatrix, then $\kappa_2(A_{11}) \leq \kappa(A)$, where κ denotes the usual 2-norm condition number for solving linear systems.

Hint: You may use the *interlace theorem*, which tells us that if $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_k$ are the eigenvalues of A_{11} and $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are the eigenvalues of A, then

$$\lambda_j \leq \mu_j \leq \lambda_{n-k+j}.$$

2 pts Show by example that the hypothesis that A is positive definite is necessary in the above statement.

2 pts Show that if $A \in \mathbb{R}^{m \times n} = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$, then $\kappa(A_1) \leq \kappa(A)$, where κ here denotes the usual condition number for least squares.