CS 4220: Final Exam

May 14, 2013

____

(Name)

____

(Cornell NetID)

Problem 1	10 points	
Problem 2	20 points	
Problem 3	15 points	
Problem 4	15 points	
Problem 5	15 points	
Problem 6	15 points	
Problem 7	10 points	

Check that your exam has ten (10) pages including this one.

1. (10 points)

(a) A 3-by-3 linear system with infinity-norm condition 10^8 is solved via the MATLAB backslash operator \ on a computer with unit roundoff 10^{-17}. Here is the computed solution:

```
xHat(1) = 1234.5678901234567

xHat(2) =    1.2345678901234567

xHat(3) =     .00012345678901234567
```

Underline the digits that are most likely correct and justify your answer by explaining how the relative error $\| \hat{x} - x \|_\infty / \| x \|_\infty$ depends on both the condition and the unit roundoff. Recall that $\| v \|_\infty = \max |v_i|$.

(b) Assume that `a` and `b` is are initialized floating point numbers with positive value and that the message "`b is small compared to a`" is displayed when the following code is executed:

```
if a + b == a

    disp('b is small compared to a')

end
```

What can you say about the actual magnitude of `b`?

2. (20 points)

(a) Show how the SVD can be used to solve the linear system

$$(A^T A + \mu I)x \; = \; A^T b$$

where $A \in \mathbb{R}^{m \times n}$, $n \leq m$, $\mu > 0$, and $b \in \mathbb{R}^m$. Answer by completing the following code in MATLAB:

```
% A, b, mu defined
[U,S,V] = svd(A);
[m,n] = size(A);
```

(b) Suppose $A = QR$ is the QR factorization of a matrix $A \in \mathbb{R}^{m \times n}$. Assume that $R(k,k) = 0$ for some $k > 1$ and that all other entries along R's diagonal are nonzero. Show how to compute a nonzero vector $x \in \mathbb{R}^n$ so that $Ax = 0$. You may use the $\backslash$ operator to solve triangular systems. Answer by completing the following code in MATLAB:

```
% A defined
[Q,R] = qr(A);
[m,n] = size(A);
```

3. (15 points) If $C \in \mathbb{R}^{n\times n}$ is symmetric and positive definite and $u \in \mathbb{R}^n$, then

$$(C + uu^T)^{-1} = C^{-1} + \alpha vv^T$$

where $v = C^{-1}u$ and $\alpha = -1/(1 + u^T C^{-1}u)$. By making effective use of the Cholesky factorization and the above math fact, complete the following function so that it performs as specified:

```
  function [x,z] = DoubleSolve(A,u,b)
% A is a symmetric positive definite n-by-n matrix.
% u and b are column n-vectors.
% x and z are column n-vectors with the property that Ax = b and
% (A + u*u')z = b.
```

You may use the \ operator to solve triangular systems.

4. (15 points) Short answer.

(a) Assume that $A \in \mathbb{R}^{n\times n}$ is symmetric with distinct eigenvalues. Describe in English what the power method would try to compute when applied to the matrix $(A - \mu I)^{-1}$.

(b) After k exact-arithmetic steps with starting unit 2-norm vector q_1 we assume that the Lanczos method computes part of the decomposition $Q^T AQ = T$ where Q is orthogonal with $Q(:,1) = q_1$ and T is tridiagonal. Explain. What makes the method a "sparse matrix friendly"? What is the method typically used for?

(c) Any symmetric positive definite matrix A has a Cholesky factorization $A = GG^T$ where G is lower triangular. Why does MATLAB sparse Cholesky software compute the factorization $PAP^T = GG^T$ where P is a permutation matrix?

5. (15 points) For each of the following methods, draw a picture that communicates the main idea behind a step. No formulas are necessary. Just a labeled sketch that graphically indicates how the next iterate is obtained. (Such a picture for Newton's method would show the linear model and label its zero.)

(a) The Secant method for finding a zero of $f: \mathbb{R} \to \mathbb{R}$.

(b) The Golden Section search method for finding a minimum of $f: \mathbb{R} \to \mathbb{R}$ on $[L, R]$ assuming that f'' is always positive.

(c) The steepest descent method with exact line search for finding a minimum of $f: \mathbb{R}^2 \to \mathbb{R}$. (Draw contours.)

6. (15 points)

(a) Consider the following MATLAB script for approximating the square root of a positive real number A:

```
L = A;
W = 1;
for k=1:10
   L = (L+W)/2;
   W = A/L;
end
```

Explain why this is essentially an instance of Newton's method.

(b) Newton's method is applied to a function $f{:}\mathbb{R} \to \mathbb{R}$ that has a single root x_*. Here is a heuristic result that relates the error at step $k+1$ to the error at step k:

$$|x_{k+1} - x_*| \;\approx\; \frac{|f''(x_k)|}{2|f'(x_k)|}|x_k - x_*|^2.$$

What additional information is required before one can be confident that the iteration converges quadratically for a particular initial guess x_0?

7. (15 points) Suppose $t, x, y \in \mathbb{R}^m$ are given and that we wish to use `lsqnonlin` to determine $a^T = [a_1, \ldots, a_k]$ and $\rho^T = [\rho_1, \ldots, \rho_k]$ so that

$$\phi(a, \rho) \;=\; \frac{1}{2}\sum_{i=1}^{m}\left(x_i - \sum_{j=1}^{k} a_j \cos\left(\frac{2\pi t_i}{\rho_j}\right)\right)^2 \;+\; \frac{1}{2}\sum_{i=1}^{m}\left(y_i - \sum_{j=1}^{k} a_j \sin\left(\frac{2\pi t_i}{\rho_j}\right)\right)^2$$

is minimized. To use `lsqnonlin` we must design a vector-valued function $R(a, \rho)$ with the property that

$$\phi(a, \rho) \;=\; \frac{1}{2} R(a, \rho)^T R(a, \rho).$$

Express $R(a, \rho)$ in the form

$$R(a, \rho) \;=\; \text{Matrix} \times \{\text{vector}\} \;-\; \{\text{vector}\}$$

Clearly define the matrix and the two vectors.

Some MATLAB Functions

LU Factorization

```
[L,U,P] = lu(X) returns unit lower triangular matrix L, upper
triangular matrix U, and permutation matrix P so that
P*X = L*U.
```

Cholesky Factorization

```
G = chol(X.lower)  returns an lower triangular G so that GG' = X
where X is symmetric and positive definite.
```

QR Factorization

```
[Q,R,E] = qr(A) produces unitary Q, upper triangular R and a
permutation matrix E so that A*E = Q*R. The column permutation E is
chosen so that ABS(DIAG(R)) is decreasing.
```

Singular Value Decomposition

```
[U,S,V] = svd(X) produces a diagonal matrix S, of the same
dimension as X and with nonnegative diagonal elements in
decreasing order, and orthogonal matrices U and V so that
X = U*S*V'.
```

Schur Decomposition for Symmetric Matrices

```
[U,D] = eig(X) produces a diagonal matrix D and
an orthogonal matrix U so that X = U*D*U' assuming that X
is real and symmetric.
```