CS 4220: Some Review Problems (Solutions/Hints)

1. Given data $(t_1, y_1), \ldots, (t_m y_m)$, show how `fmninbnd` can be used to solve the problem of minimizing α_1, α_2 and λ so that

$$\phi(a,b,\lambda) \;=\; \sum_{i=1}^{m} \left(\alpha_1 + \alpha_2 e^{\lambda t_i} - y_i\right)^2 ; = \; \| \, A(\lambda)\alpha - y \, \|_2^2$$

is minimum?. What is $A(\lambda)$? What is the best choice of α given λ? What is the function passed to `fminbnd`?

```
% One possibility...
function z = MyF(lambda,t,y)
m = length(y);
A = [ ones(m,1) exp(lambda*t)];
[Q,R] = qr(A);
% The min value of norm(A*alfa - y)*2 is norm(Q2'*y)^2 where
% Q2 = Q(:,3:m).
z = norm(Q(:,3:m)'*y)^2

% Would need to supply a search interval to fminbnd.

% is it possible to compute z using QR(A,0) ?
```

2. Assume the availability of the following function:

```
  function [Q,R] = Update(Q0,R0,u,v)
% Q0 is m-by-m orthogonal, R0 is m-by-n upper triangular, u is m-by-1, v is n-by-1
% Q is m-by-m orthogonal, R is m-by-n upper triangular, Q*R = Q0*R0 + u*v'
```

(a) How would you solve min $\| \, Ax - b \, \|$ and min$\| \, \tilde{A}x - b \, \|$ given that $\tilde{A}$ is defined by $\tilde{A}(:,k) = A(:,k) - sum(A(:,k))$.

(b) How would you solve min $\| \, Ax - b \, \|$ and min$\| \, \tilde{A}x - b \, \|$ given that $\tilde{A}$ is A with a new row appended to the bottom.

(c) How would you solve min $\| \, Ax - b \, \|$ and min$\| \, \tilde{A}x - b \, \|$ given that $\tilde{A}$ is A with its first column replaced by a given vector.

```
% (a)
  [m,n] = size(A);
  [Q0,R0] = qr(A);
  x = R0(1:n,1:n)\(Q0(:,1:n)'*b)
  v = sum(A)';   % i.e., v(j) = sum(A(:,j)), j=1:n
  u = -ones(m,1)
  [Q,R] = update(Q0,R0,u,v);
  xtilde  = R(1:n,1:n)\(Q(:,1:n)'*b)
```

```
% (b)
  [m,n] = size(A)
  % adding a zero row to A and zero to b does not change the solution to the LS prob...
  A = [A; zeros(1,n)]; b = [b;0];
  [Q0,R0] = qr(A);
  x = R0(1:n,1:n)\(Q0(:,1:n)'*b)
  u = [zeros(m,1);1] % The last column of eye(m+1,m+1)
  % Assume that v' is the row to be appended
  [Q,R] = Update(Q0,R0,u,v)      % Atilde = [A;v']
  xtilde  = R(1:n,1:n)\(Q(:,1:n)'*b)

% (c)
  % Atilde = A + (newCol - A(:,1)e1'  where e1  first col of eye(n,n)
   [m,n] = size(A);
   [Q0,R0] = qr(A);
   x = R0(1:n,1:n)\(Q0(:,1:n)'*b);
   % Assume neCol is the new column 1..
   u = newCol - A(:,1);
   v = [1;zeros(n-1,1)]
   [Q,R] = Update(Q0,R0,u,v)
   xtilde  = R(1:n,1:n)\(Q(:,1:n)'*b)
```

3, $M \in \mathbb{R}^{n\times n}$ has the property that the first $n-1$ columns of $M - I_n$ are independent. How would you compute a nonzero vector x so $Mx = x$ given that $P(M-I) = LU$ is on tap?

```
% Assume M - I is singular. (FORGOT TO MENTION THIS!)
[L,U,P] = lu(M - eye(n,n));
% P(M - I) = LU. Since M-I is singular, U must be singular.
% So if we can find a nonzero x so Ux = 0, it follows that P(M-I)x = 0,
% and so (M-I)x = 0, i.e., Mx = x.
% Since the first n-1 columns of M-I are independent, the same can
% be said about the first n-1 columns of inv(L)P(M-I) = U. So we must
% have U(n,n) = 0. Set x(n) = -1. From
%
%            U(1:n-1,1:n-1) U(1:n-1,n)    x(1:n-1)      0
%                                        *             =
%                  0              0       -1            0


  x = zeros(n,1); x(n) = -1; x(1:n-1) = U(1:n-1,1:n-1)\U(1:n-1,n)
```

4. $G \in \mathbb{R}^{n\times n}$ is a 0-1 matrix in sparse format. Define the matrix $B \in \mathbb{R}^{n\times n}$ by

$$B(:,k) \;=\; \begin{cases} G(:,k)/sum(G(:,k)) & \text{if } G(:,k) \neq 0 \\ \\ \text{ones}(n,1)/n & \text{otherwise} \end{cases}$$

Let $0 < \rho < 1$ be a scalar. How would you apply the power method to compute the dominant eigenvector for the matrix $A = \rho B + (1-\rho) * \text{ones}(n,n)/n$. Show carefully how you would organize the matrix-vector products.

```
 % We say v is a probability vector if v(i)>= 0 all i, and sum(v) = 1.
 % Power method step:
 %   x = A*x ; x = x/norm(x,1)
 % We use the 1-norm because if x is a probability vector then Ax is a
```

```
% probability vector.
% Let e = ones(n,1)
%
%  Ax = rho*B*x + (1-rho)(ones(n,n)/n)*x
%     = rho*B*x + ((1-rho)/n)ee'*x
%     = rho*B*x + ((1-rho)/n)*e
%     = rho*(G(:,ip)*(x(ip)./d(ip)) + sum(x(iz))e/n) + ((1-rho)/n)*e
_
% where
  d = sum(G); iz = d==0; ip = d>0;

% Precompute these vectors...
  dtilde = d(ip)/rho;
  etilde1 = (rho/n)*ones(n,1);
  etilde2 = ((1-rho)/n)*ones(n,1)


% How to compute y = A*x
  y = G(:,ip)*(x(ip)./dtilde) + sum(x(iz))*etilde1 + etilde2
```

5. $Q \in \mathbb{R}^{n\times n}$ is orthogonal with a unique eigenvalue equal to one. What happens if the power method is applied to the matrix $A = I + (Q + Q^T)/2$? Hint: Draw a picture of A's eigenvalues.

```
%All the eigenvalues of Q are on the unit circle. (Take 2-norms in Qx = lambda*x.)
% If Qx = lambda*x then
% Ax = x + .5*(Qx + Q'x) = x + .5*(lambda*x + 1/lambda)*x
%                        = x + .5*(lambda + conj(lambda))*x
%                        = (1 + real(lambda))x
% so all the eigenvalues of the symmetric matrix A are in the interval [0,2]
% The eigenvalue 2 is unique. So the power method finds an x so Qx = x.
```

6. (a) How would you minimize $\| Ax \|_2$ subject to the constraint that $x_1 = 1.$? (b) Given $t \in \mathbb{R}^m$, how would you determine a scalar τ so that

$$\sum_{k=1}^{m} [\sin(t_i - \tau)]^2$$

is minimized? Hint: Trig identities and SVD.

```
% (a)
% min norm(Ax) = min norm(A(:,1) + A(:,2:n)x(2:n)))
 xopt = [1; -A(:,2:n)\A(:,1)]

% (b)
% Remembering that t is a column vector,
% The summation = norm(sin(t - tau))^2
%               = norm(sin(t)*cos(tau) - cos(t)*sin(tau))^2
%               = norm([sin(t) -cos(t)]*[cos(tau);sin(tau)])^2
% We want a unit vector v = [cos(tau);sin(tau)]  so that
%         norm(Av) = min where A = [sin(t) -cos(t)], an m-by-2 matrix.
% That is an SVD problem..
  [U,S,V] = svd([sin(t) -cos(t)])
  v = V(:,2)
  tau = atan2(v(2),v(1))
```

7. What does a step of steepest-descent with exact line search look like when applied to the minimization of $\phi(x) = x^T Ax/x^T x$ where $A \in \mathbb{R}^{n\times n}$ is symmetric?

```
% The gradient...
  g = (2(x'*x)*A*x - 2(x'*A*x)x)/(x'*x)^2 = 2(A - phi(x)I)x;
% Exact line search requires finding mu so that f(mu) = phi(x - mu*g) is minimum
% Solve f'(mu_opt) = 0  for the optimizing mu
```

8. Newton's method is applied to find a, b, and c so that $(x-a)(x-b)(x-c) = x^3 + 5x^2 - 3x + 4$. Describe the linear system that must be solved each step.

```
% (x-a)(x-b)(x-c) = (x^2 -(a+b)x +ab)(x-c)
%                  = x^3 - x^2(a+b+c) +(ab + bc + ca)x -abc
% Find a zero of
%                 a+b+c+5
%    F(a,b,c) =  ab+bc+ca+3
%                 abc+4
%
%                  1     1     1
%    Jacobian =   b+c   a+c   a+b
%                  bc    ac    ab
```

9. How could Householder tridiagonalization be used to solve the linear system $(A^2 - 5A + 6I)x = b$ where $A \in \mathbb{R}^{n\times n}$ is symmetric?

```
%   Solve (A-2I)(A-3I)x = b.
%   Solve (A-2I)y = b for y, and (A-3I)x = y for x.
%   So, Compute the tridiagonalization A = QTQ' and solve
%      Q(T-2I)Q'y = b  for y: y = Q*((T-2I)\(Q'*b))
%      Q(T-3I)Q'x = y  for x: x = Q*((T-3I)\(Q'*y))
%   Note that T-2I and T-3I are tridiagonal.
```

10. Suppose $A, B \in \mathbb{R}^{m\times n}$ and $m < n/2$. How could you find $x \in \mathbb{R}^n$ so that $Ax = b$ and $Bx = c$ where $b, c \in \mathbb{R}^m$ are given?

```
% We must solve the 2m-by-n underdetermined system [A;B]x = [b;c]
M = [A;B]; d = [b;c];
% Solve Mx = d...
[Q,R] = qr(M')
% Mx = d implies (R'Q'x = d for x
z = R(1:2m,1:2m)'\d
x = Q(:,1:2m)*z
```