CS 422: Prelim I Solutions

1. (20 points)

(a) Compute an *upper* triangular matrix $U \in \mathbb{R}^{2\times 2}$ so that

$$\begin{bmatrix} 5 & 10 \\ 10 & 25 \end{bmatrix} = UU^T$$

Solution

$$\begin{bmatrix} 5 & 10 \\ 10 & 25 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

implies that $c^2 = 25$ (so $c = 5$), $bc = 10$ (so $b = 2$), and $a^2 + b^2 = 5$ (so $a = 1$). (10 points)

(b) A small relative change in b can induce a relatively large change in x when solving a linear system $Ax = b$. Explain this statement using the singular value decomposition. *Solution*

$$x = A^{-1}b = (U\Sigma V^T)^{-1}b = V\Sigma^{-1}U^Tb = \sum_{i=1}^{n} \frac{u_i^T b}{\sigma_i} v_i$$

where $U = [u_1 \ \cdots \ u_n]$, $V = [v_1 \ \cdots \ v_n]$, $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n$. If b is changed a small amount and the change is not orthogonal to u_n, then those changes are magnified by $1/\sigma_n$. (10 points). You cannot just say that if A is ill-conditioned then we have solution sensitivity. That is just restating the *definition* of ill-conditioning. Up to 5 points for intelligent (but irelevant) SVD facts.

2. (20 points)

A $2n$-by-$2n$ matrix of the form

$$M = \begin{bmatrix} A & F \\ G & -A^T \end{bmatrix}$$

is a *Hamiltonian* matrix if $A, F, G \in \mathbb{R}^{n\times n}$ and F and G are symmetric. Write a flop-efficient MATLAB function of the form `Y = HamProd(A,F,G)` that assigns to `Y` the $2n$-by-$2n$ matrix M^2, square of the Hamiltonian matrix defined by A, F, and G. No loops are necessary.

Solution

$$\begin{bmatrix} A & F \\ G & -A^T \end{bmatrix}^2 = \begin{bmatrix} A^2 + FG & AF - FA^T \\ GA - A^TG & (A^T)^2 + GF \end{bmatrix}$$

So,

```
M1 = A*A + F*G;
M2 = A*F;
M3 = G*A;
Y = [ M1  (M2-M2') ; (M3-M3')   M1']
```

3. (20 points)

Complete the following function so that it performs as specified

```
  function C = HouseUpdate(A,r,u)
% A is an m-by-n matrix.
% 1<=r<m
% u is a column m-vector with u(r+1:m) = 0
% C = H*A where H = I - 2*u*u'
```

No loops are necessary.

Solution

```
v = u(1:r);
C1 = A(1:r,:) - (2*v)*(v'*A(1:r,:))    % 10 points, -5 if parens wrong, -10 if form Householder
C = [C1 ; A(r+1:n,:)];           %10 points, -5 if you set C(r+1:m,:) = 0
```

4. (20 points)

The Sherman-Morrison formula states that

$$\left(A + fg^T\right)^{-1} \;=\; A^{-1} + \alpha c d^T$$

where $A \in \mathbb{R}^{n\times n}$, $f, g \in \mathbb{R}^n$, $c = A^{-1}f$, $d^T = g^T A^{-1}$, and $\alpha = -1/(1+g^T A^{-1} f)$. It is assumed that both A and $A + fg^T$ are nonsingular. Complete the following function so that it performs as specified

```
  function [x,y] = Solve(A,b,f,g)
% A is n-by-n and nonsingular. b,f, and g are column n-vectors and
% A + f*g' is nonsingular
% x and y are column n-vectors that satisfy A*x = b and (A + f*g')y = b.
```

Solution

If $x = A^{-1}b$ and $c = A^{-1}f$ then

$$y = x - \frac{g^T x}{1 + g^T c} c$$

Thus,

```
[L,U,P] = lu(A)          % -3 if no pivoting
x = U\(L\(P+b));
c = U\(L\(P*f));
y = x - ((g'*x)/(1+g'*c))*c
```

5. (20 points)

Suppose you have the "thin" QR factorization of $A \in \mathbb{R}^{m\times n}$ and $\mathrm{rank}(A) = n$. Suppose $B \in \mathbb{R}^{m\times N}$ is a given matrix. Show how to efficiently compute the numbers $\rho_1, \ldots, \rho_N$ where ρ_j is the minimum possible value of $\| Ax - B(:,j) \|_2$, $j = 1{:}N$. A MATLAB fragment is not necessary but you may present your solution in that format if you wish.

Solution

If $A = QR$ is the full QR factorization, and $R_1 = R(1{:}n, 1{:}n)$, $Q_1 = Q(:, 1{:}n)$ and $Q_2 = Q(:, n+1{:}m)$, then $\| Ax - b \|_2^2 = \| R_1 x - c \|_2^2 + \| d \|_2^2$ where $c = Q_1^T b$ and $d = Q_2^T b$. Thus, $x_{LS} = R_1 \backslash (Q_1^T b)$ solves $\min \| Ax - b \|$ and the minimum sum of squares is $\| d \|_2^2$. Since you only have the factorization $A = Q_1 R_1$, use the fact that $\| b \|_2^2 = \| c \|_2^2 + \| d \|_2^2$ and so $\rho_i = \sqrt{\| B(:,i) \|_2^2 - \| Q_1^T B(:,i \|_2^2}$. Minus 2 for $\rho_i = \| A * (R_1 \backslash (Q_1^T b)) - b \|_2$. (More work)