CS 421: Numerical Analysis
Fall 2005
Prelim 1

Handed out: Thurs., Sep. 22.
This exam has five questions. You have 75 minutes to answer all the questions. Write your answers in the booklet. You may consult a $8.5'' \times 11''$ piece of paper written on both sides that you have prepared in advance. The questions are weighted equally even though they are not equally difficult.

1. Consider the easily verified algebraic identity:

$$\frac{1}{x-1} - \frac{1}{x} = \frac{1}{x(x-1)}.$$

   Direct evaluation of one side of this equation is more prone to catastrophic cancellation than the other when $x \gg 1$. Explain.

2. Consider the problem of evaluating the inner product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$. Give an example of an ill-conditioned instance of this problem and also give an example of a well-conditioned instance. Note: Interpret this question in the relative sense, i.e., find an instance so that a small relative perturbation to $\mathbf{x}$ or $\mathbf{y}$ causes either a small or large relative perturbation to $\mathbf{x}^T\mathbf{y}$. Explain your answers. [Hint: The deciding factor is whether $\mathbf{x}^T\mathbf{y}$ is close to zero.]

3. Let A be an $n \times n$ matrix whose second column is a multiple of its first column. Assume neither column is all-zero. Show that GEPP will terminate with a '0' pivot on the second elimination step.

4. (a) Construct a 2×2 matrix A such when GEPP is applied to this matrix, there is underflow in the computation of multiplier $L(2,1)$. Recall that underflow means that the result has magnitude less than about 10^{-308} in IEEE double precision but is not zero.

   (b) Suppose that this underflowing multiplier $L(2,1)$ gets changed to 0. Argue that this change does not harm the stability of GEPP. [Suggested approach: Explain why changing the underflow to 0 is mathematically equivalent to making a perturbation E to your original matrix A such that $\|E\| \ll \|A\|$.]

5. Let A be an $n \times n$ nonsingular matrix. Show that the function $\|\cdot\|_A$ defined by $\|\mathbf{x}\|_A = \|A\mathbf{x}\|_2$ is a norm on $\mathbf{R}^n$ (i.e., that it satisfies the three axioms of a norm).