CS 421: Numerical Analysis
Fall 2004
Problem Set 5

Handed out: Wed., Nov. 10.

Due: Fri., Nov. 19 in lecture.

1. The *Krylov space* $K_k(A, \mathbf{b})$ for an $n \times n$ matrix A and n-vector $\mathbf{b}$ is defined to be

$$K_k(A, \mathbf{b}) = \mathrm{Span}(\mathbf{b}, A\mathbf{b}, A^2\mathbf{b}, \cdots, A^{k-1}\mathbf{b}).$$

   (a) Argue by induction that $\mathbf{x}^{(k)}$ computed by conjugate gradient lies in $K_k(A, \mathbf{b})$. (Assume the starting guess is $\mathbf{x}^{(0)} = \mathbf{0}$.)

   (b) Show that the solution $\mathbf{x}$ to the linear system $A\mathbf{x} = \mathbf{b}$ lies in $K_n(A, \mathbf{b})$. Assume A is symmetric and positive definite. [Hint: Clearly the $n+1$ vectors $\mathbf{b}, A\mathbf{b}, \ldots, A^n\mathbf{b}$ must be linearly dependent since they lie in $\mathbf{R}^n$. Write out an equation of linear dependence, and pay attention to the index i such that the coefficient of $A^i\mathbf{b}$ is nonzero, and such i is minimal with this property.]

2. Consider finding an real eigenpair of a matrix $A \in \mathbf{R}^{n\times n}$. This can be accomplished by solving the system of $n+1$ nonlinear equations $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x}^T\mathbf{x} = 1$ for the $n+1$ variables $(\mathbf{x}, \lambda)$.

   (a) Write out Newton's method for these nonlinear equations.

   (b) Show how a preliminary Hessenberg factorization of A can reduce the number of flops need per Newton iteration.

3. It has been proposed in the literature to use Newton's method to compute the inverse of a matrix. Let A be an $n \times n$ nonsingular matrix. Consider the nonlinear equations $f(X) = A - X^{-1}$. Then if $f(X^*) = 0$, clearly $X = A^{-1}$. It can be shown that the Newton iteration for solving $f(X) = 0$ is $X^{(k+1)} = 2X^{(k)} - X^{(k)}AX^{(k)}$.

   (a) Show by a direct argument that this iteration converges quadratically provided that all the eigenvalues of $AX^{(0)} - I$ are less than 1 in absolute value. [Hint: Let $Y^{(k)} = AX^{(k)} - I$. Find a formula for $Y^{(k+1)}$ in terms of $Y^{(k)}$.]

   (b) Show that there exists an $\alpha > 0$ such using αA^T for $X^{(0)}$ satisfies the condition in (a) (i.e., for this particular $X^{(0)}$, all eigenvalues of $AX^{(0)} - I$ are less than 1 in absolute value). Note that solving part (b) of this question does not require knowing how to solve part (a).

4. Implement the method for matrix inversion described in the previous question. Take $\alpha = 1/\|A\|_F^2$ for the starting point described by (b). Let

$$\rho_k = \|AX^{(k)} - I\|_F / (\|A\|_F \cdot \|X^{(k)}\|_F).$$

Note: $\|A\|_F$ in Matlab is `norm(A,'fro')`. Use as a termination test $\rho_k \leq \epsilon_{\text{mach}}$. Use also as a second termination test $\rho_k \geq 0.99\rho_{k-1}$ (to detect stagnation of the iteration).

Determine how many iterations are required for convergence, as a function of the condition number of A. Use a very wide range of condition numbers. Note: Produce matrices with a given value of condition number by forming a diagonal matrix with known condition number, and then multiply on the left and right by a random orthogonal matrix. A random orthogonal matrix can be generated by applying `qr(C)` to a random matrix C. Use `semilogx` for plotting so that the condition number axis uses a log-scale.

Plot also the final residual ρ_k (i.e., the residual at the step that the termination test is satisfied) as a function of condition number. Use `loglog` plots for this purpose.

Finally, plot the same residual for the same set of matrices, except use the inverse as computed by the built-in `inv(A)`.

Hand in listings of your m-files and the requested plots plus a paragraph of conclusions.