CS 421: Numerical Analysis
Fall 2004
Practice Prelim 1

Handed out: Wed., Sep. 15 (web only).

This test lasted 75 minutes. All the questions were weighted equally even though they are not equally difficult. Students were allowed to consult a 8.5-by-11 sheet of paper that they had prepared in advance.

1. Let $\mathbf{x}$ be a vector in $\mathbf{R}^n$. (a) Show that

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \leq n\|\mathbf{x}\|_\infty.$$

   (b) Exhibit two nonzero vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^n$ such that the first inequality of part (a) is tight (i.e., is satisfied as an equation) for $\mathbf{x}_1$, while the second inequality is tight for $\mathbf{x}_2$.

2. Consider the function $f(x) = \cos x - 1$. (a) Show that the obvious way for evaluating this function is prone to catastrophic cancellation for x close to 0. (b) Propose an alternative way to evaluate this function when x is close to 0. [Hint for (b): recall $\cos 2\theta = \cos^2\theta - \sin^2\theta$.]

3. *Threshold pivoting* is a strategy sometimes used in place of partial pivoting within Gaussian elimination applied to an $n \times n$ matrix A. In threshold pivoting, any uneliminated entry $A(p,k)$ in the pivot column k may be selected as pivot provided $|A(p,k)| \geq \alpha \max|(A(k:n,k))|$ where α is a parameter between 0 and 1. (For example, $\alpha = 1$ would be partial pivoting.) Assuming threshold pivoting is used, derive an upper bound on $\|L\|_\infty$ in terms of n and α, where L is the lower triangular factor resulting from elimination.

4. Consider the problem of evaluating a real-valued differentiable function $f(x)$ of a scalar variable x. The condition number of this problem for argument x_1 is sometimes defined to be $|f'(x_1) \cdot x_1|/|f(x_1)|$. Explain why. [Hint: consider small relative perturbations to x_1. The derivative comes from a Taylor approximation.]

5. Suppose Gaussian elimination with pivoting is performed on a nonsingular $2n \times 2n$ matrix with block structure

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

   where all the blocks are size $n \times n$. Show that both factors L and U will have a block of zero entries, and determine the number of flops (accurate to the leading term) for computing the P^TLU factorization of this matrix.