CS 421: Numerical Analysis
Fall 2004
**Practice Final Exam**

Handed out: Tues., Dec. 7 (web only).

This exam lasted 120 minutes. Students were allowed to consult a prepared sheet of notes (one page, $8\frac{1}{2}'' \times 11''$ written on both sides).

1. [**5 points**] How many flops (accurate to the leading term) are required to apply a Givens rotation to two vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$?
2. [**5 points**] Same as the previous question, except assume further that $\mathbf{y} = \mathbf{e}_1$ (where $\mathbf{e}_1$ is the first column of the identity matrix).
3. [**5 points**] Let $A$ be a square nonsingular upper triangular matrix. If Gaussian elimination is applied to $A$, then the same factorization results regardless of whether partial pivoting is used or not. Explain why.
4. [**5 points**] Let $A$ be a square nonsingular lower triangular matrix. If Gaussian elimination is applied to $A$, is the LU factorization different depending on whether partial pivoting is used?
5. [**5 points**] According to the Cornell Trustee report for selection of the new Cornell President, two out of the following three items are challenges faced by the next president: (1) Build in strategic sciences, (2) Enhance humanties, arts and social sciences, and (3) Develop an $O(n^2)$ algorithm for solving $A\mathbf{x} = \mathbf{b}$. Which one of these three is not mentioned by the trustees? And why not??
6. [**10 points**] Let $\mathbf{x}^*$ be the solution to the full-rank linear least squares problem of minimizing $\|A\mathbf{x} - \mathbf{b}\|_2$. Using the system of normal equations, show that $A\mathbf{x}^*$ is orthogonal to $A\mathbf{x}^* - \mathbf{b}$.
7. [**10 points**] Let $U$ be an $n \times n$ upper triangular matrix of rank $r$ such that exactly $p$ of its diagonal entries are nonzero. Under these assumptions, there are several inequalities involving $n, r, p$ that must always hold. Write them all down.
8. [**10 points**] Consider finding $\mathbf{x} \in \mathbf{R}^n$ to minimize $\|A_1\mathbf{x} - \mathbf{b}_1\|_2^2 + \alpha\|A_2\mathbf{x} - \mathbf{b}_2\|_2^2$, where $\alpha > 0$, $A_1 \in \mathbf{R}^{p\times n}$, $A_2 \in \mathbf{R}^{q\times n}$, $\mathbf{b}_1 \in \mathbf{R}^p$, $\mathbf{b}_2 \in \mathbf{R}^q$. All of these quantities $(\alpha, A_1, A_2, \mathbf{b}_1, \mathbf{b}_2)$ are given as problem data. Rewrite this problem as a standard linear least-squares problem, and determine its system of normal equations.
9. [**15 points**] Suppose the power method is applied to an $n \times n$ diagonalizable matrix $A$ whose eigenvalues are given by $\lambda_1, \ldots, \lambda_n$. Suppose these eigenvalues satisfy

   $$|\lambda_1| = |\lambda_2| > |\lambda_3| \geq \cdots \geq |\lambda_n|.$$

   Suppose further that $\lambda_1 = \lambda_2$. Under these assumptions, would you expect the power method to converge to an eigenvector? Explain.

10. **[25 points]** Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a nonlinear function and $\mathbf{x}^{(0)}$ an initial point, and suppose we wish to find a point $\mathbf{x}^*$ such that $f(\mathbf{x}^*) = \mathbf{0}$ using Newton's method. It is sometimes possible to "left-precondition" Newton's method, that is, to find a function $g : \mathbf{R}^n \to \mathbf{R}^n$ with the property that $g(\mathbf{0}) = \mathbf{0}$, and then solve $h(\mathbf{x}) = \mathbf{0}$ using Newton's method (rather than $f(\mathbf{x}) = \mathbf{0}$), where $h(\mathbf{x})$ is defined to be $h(\mathbf{x}) = g(f(\mathbf{x}))$.

   (a) Suppose the left preconditioner is linear, i.e., suppose $g(\mathbf{y}) = A\mathbf{y}$ where $A$ is an invertible matrix. Show that in this case, the left-preconditioned Newton iteration produces the same sequence of $\mathbf{x}^{(k)}$ as the original.

   (b) On the other hand, come up with an example in which nonlinear left-preconditioning leads to a big improvement. [Hint: there are examples with $n = 1$ in which the left-preconditioned system converges in a single step.]

11. **[25 points]** Consider a system of ODEs with two scalar unknown functions $v(t)$ and $x(t)$ that are governed by equations of the form $dv/dt = f(x)$ and $dx/dt = g(v)$. (This is called a "partitioned problem.") One finite difference algorithm for solving this problem (related to the "leapfrog Verlet method") is given by

$$\begin{aligned} v_{k+1} &= v_k + hf(x_k), \\ x_{k+1} &= x_k + hg(v_{k+1}). \end{aligned}$$

   where $h$ is the stepsize.

   (a) What is the distinction between this method and the Euler method?

   (b) Show that this method, like Euler, has order of accuracy equal to 1.

   (c) Suppose this method is applied to the specific problem $dv/dt = -x$, $dx/dt = v$, which has as its exact solution $x(t) = \alpha \sin t + \beta \cos t$, $v(t) = \alpha \cos t - \beta \sin t$, where $\alpha, \beta$ depend on the initial conditions. Show that the finite difference method above yields a recurrence of the form $\mathbf{y}_{k+1} = A\mathbf{y}_k$ where $\mathbf{y}_k$ stands for $(v_k, x_k)$ and $A$ is a fixed matrix.

   (d) Continuing from the assumptions in part (c), determine a limit on the stepsize $h$ to ensure there is no exponential growth in the computed solution, and explain your answer.