CS 421: Numerical Analysis
Fall 2002
Problem Set 6

Handed out: Mon., Nov. 20.

Due: Fri., Dec. 6 in lecture.

1. (a) (From Trefethen's book on the web.) Consider the IVP $dy/dt = y^{1/2}$, $y(0) = 0$. Show that this IVP has two different solutions of the form $y = at^b$ for constants a and b. Then show that in fact the IVP has an infinite number of solutions that can be obtained by gluing these two solutions together.

   (b) Look up a "uniqueness" theorem for the solution of IVP's, e.g., read p. 387 of the text. Explain why the uniqueness theorem does not apply to this IVP.

2. For both parts of this question, assume constant stepsize.

   (a) Determine the local truncation error (including the coefficient) of 2-step Adams Moulton rule, which is

   $$y_{k+1} = y_k + (5h/12)f(y_{k+1}, t_{k+1}) + (8h/12)f(y_k, t_k) - (h/12)f(y_{k-1}, t_{k-1})$$

   using the R-method from lecture.

   (b) Show that the order of accuracy of the method $y_{k+1} = 2y_k - y_{k-1}$ is 1 (i.e., same order as EM). There is something obviously wrong with this method! Explain why this method is completely useless in practice.

3. Consider a frictionless pendulum, whose equations of motion are:

   $$\begin{aligned}\frac{d\theta}{dt} &= v,\\ \frac{dv}{dt} &= -\sin\theta\end{aligned}$$

   where θ is the angle made by the pendulum with respect to vertical and v is the angular velocity of the pendulum.

   (a) Verify mathematically that this system conserves energy, where energy is $E = -\cos\theta + v^2/2$. [Hint: Compute dE/dt.]

   (b) Show that for Euler's method applied to the pendulum, $E_{k+1} = E_k + O(h^2)$. Here, E_k stands for $-\cos\theta_k + (v_k)^2/2$. Assuming the swings are small (θ_k fairly close to 0) and h is small, is $E_{k+1} - E_k$ positive or negative? [Hint: Expand $E_{k+1} - E_k$ as a Taylor series in h. Show that the $O(h)$ term vanishes, and study the coefficient of the $O(h^2)$ term to decide whether it will be positive or negative.]

4. Implement AB1 (i.e. Euler's method) and AB2 for the pendulum problem of Q3 in Matlab and track the energy of the system. Choose starting data $\theta_0 = \pi/4$ and $v_0 = 0$ (i.e., the pendulum is stationary and at a 45-degree angle). How well do each of them conserve energy for a fixed time step and fixed interval of integration? What happens to each when the time step is halved (but the interval of integration is fixed)? Note that you can initialize AB2 by taking one step of AB1 to start.

   Run AB1 for a very large number of steps. You will notice there is eventually a qualitative transition to a different kind of behavior. Can you explain this transition? (The same thing will happen to AB2, if the number of steps is large enough.)

   Turn in listings of your m-files, a paragraph or two of conclusions and at least one interesting plot.