CS 421: Numerical Analysis
Fall 2002
Problem Set 5

Handed out: Wed., Nov. 13.

Due: Fri., Nov. 22 in lecture.

1. Consider finding the root of the equation $\arctan(x) = 0$ using Newton's method. (Note that this equation has a unique root $x^* = 0$.) Partition $\mathbf{R}$ into three subsets A, B, C such that (a) for any $x^{(0)} \in A$, the sequence $x^{(k)}$ generated by NM converges to x^*, (b) for any $x^{(0)} \in B$, $|x^{(k)}|$ tends to ∞, (c) for any $x^{(0)} \in C$, $x^{(k)}$ does not tend to 0 and also does not grow without bound.

   [Hint: to get started, draw a plot.]

2. Following the hypothesis and proof style of the theorem in lecture, show that for NM, as $k \to \infty$ and $x^{(k)} \to x^*$, then $|f(x^{(k)})|$ is decreasing with at least a linear rate.

   [Hint: Derive the identity

   $$f(x^{(k)}) = \int_{x^*}^{x^{(k)}} (f'(t) - f'(x^*))\,dt + \int_{x^*}^{x^{(k)}} f'(x^*)\,dt.$$

   Use the Lipschitz bound on the first term to show that its absolute value is dominated by the second term.]

3. It has been proposed in the literature to use Newton's method to compute the inverse of a matrix. Let A be an $n \times n$ nonsingular matrix. Consider the nonlinear equations $f(X) = A - X^{-1}$. Then if $f(X^*) = 0$, clearly $X = A^{-1}$. It can be shown that the Newton iteration for solving $f(X) = 0$ is $X^{(k+1)} = 2X^{(k)} - X^{(k)}AX^{(k)}$.

   (a) Show by a direct argument that this iteration converges quadratically provided that all the eigenvalues of $AX^{(0)} - I$ are less than 1 in absolute value. [Hint: Let $Y^{(k)} = AX^{(k)} - I$. Find a formula for $Y^{(k+1)}$ in terms of $Y^{(k)}$.]

   (b) Show that there exists an $\alpha > 0$ such using αA^T for $X^{(0)}$ satisfies the condition in (a) (i.e., for this particular $X^{(0)}$, all eigenvalues of $AX^{(0)} - I$ are less than 1 in absolute value). Note that solving part (b) of this question does not require knowing how to solve part (a).

4. The *basin of attraction* for a root $\mathbf{x}^*$ of $f(\mathbf{x}) = \mathbf{0}$ in Newton's method is defined to be the set of points $\mathbf{x}^{(0)}$ such that NM will converge to $\mathbf{x}^*$ starting from $\mathbf{x}^{(0)}$. Sometimes basins of attraction have a simple structure as in Q1, but sometimes their structure is very complex and interesting.

   Consider the polynomial equation $z^3 - 1 = 0$, which has three roots in the complex plane, namely, 1, $-1/2 + \sqrt{3}i/2$, $-1/2 - \sqrt{3}i/2$. Implement a NM for this equation using Matlab's complex arithmetic. Search over a fine grid of points covering the disk of radius 3 centered at 0 in the complex plane, and check which ones are in the basin of each of these three roots. (Some points may not be in any basin.) Then make a printout in which you color-code points in the disk according to which basin they are in. (Use red for the first basin, blue for the second, and green for the third, with black representing a point not in any basin. If you do not have access to a color printer, use a grayscale encoding.) There is work in the literature by several groups, including Kozen and Stefansson from Cornell, concerning the structure of basins for Newton's method.

Some issues: for efficiency, you don't need to wait until NM has "converged" to the root in question. Instead, it suffices that NM is close enough to the root, and then you can terminate the iteration and colorcode the starting point. For "close enough", you can invoke the theorem in lecture. (The theorem generalizes to complex numbers, essentially without change.) Similarly, for points that are converging to infinity, you can simply check whether the absolute value exceed some certain cutoff C, then NM is certain to diverge. (To find C, take common denominators; the numerator in NM is cubic and denominator quadratic, so the numerator is bound to dominate for $|z|$ sufficiently large.)

Hand in listings of m-files, the requested printout, and a paragraph of conclusions. Indicated how you addressed the issues in the last paragraph.