CS 421: Numerical Analysis
Fall 2002
Prelim 2

Handed out: Friday., Nov. 8.

This exam has four questions. You have 72 hours to answer all questions. The exam is due in lecture on Monday, Nov. 11. The questions are weighted equally even though they are not equally difficult. The exam counts for 20% of your final course grade (same as Prelim 1).

The exam is open-book and open-note. You may also consult outside published sources. If you use material from sources other than Heath and the lecture notes, then you must cite your sources.

You may also consult the web and other on-line resources. But you may not make any posting or send any email concerning the exam questions.

Academic integrity. You are not allowed to collaborate at all on this exam. You should refrain from bringing up the questions in any kind of discussion until the afternoon of Fri., Nov. 15 because some students may be handing in the exam late. The following kind of cooperation is allowed: You are permitted to borrow and photocopy someone else's lecture notes or other publicly-available course material. Please write and sign the following statement on your solutions: "I have neither given nor received unpermitted assistance on this exam."

Help from the instructor. The only help available will be clarification of the questions. No help will be given towards finding a solution.

Late acceptance policy. Solutions turned in up to 24 hours late will be accepted with a 10% penalty. The full late penalty is applied even if a portion of the solutions are returned on time. No solutions will be accepted more than 24 hours late.

1. Let H be an affine subset of $\mathbf{R}^n$, and let $\mathbf{w} \in \mathbf{R}^n$ be a given point.

   (a) Provide an algorithm based on QR factorization to find the point $\mathbf{y} \in H$ closest to $\mathbf{w}$ in the 2-norm assuming that H is presented in parametric form $H = \{A\mathbf{u}+\mathbf{b} : \mathbf{u} \in \mathbf{R}^k\}$. (See PS3 for the relevant definitions.) Analyze the number of flops to find $\mathbf{y}$ accurate to the leading terms in n, k where k is the dimension of the affine space.

   (b) Then repeat everything in (a) for the case that H is presented in implicit form $H = \{\mathbf{x} : B\mathbf{x} = \mathbf{c}\}$.

   [Hint: Part (a) is more straightforward than part (b). For part (b), the relevant algorithm in the special case $\mathbf{w} = \mathbf{0}$ is given by Golub and Van Loan on p. 272 and is repeated as an appendix at the end of this prelim. You can reduce the general case when $\mathbf{w}$ is nonzero to this special case via an appropriate change of variables.]

2. Propose an $O(n^2)$ algorithm for the following problem, and analyze the running time of your proposed algorithm accurate to the leading term. Given a unit vector $\mathbf{q}_1 \in \mathbf{R}^n$,

determine a lower Hessenberg orthogonal matrix Q such that $Q(:,1) = \mathbf{q}_1$. The matrix Q must be computed in explicit form by the algorithm.

[Hint: Consider applying Givens rotations to $\mathbf{q}_1$ in the order $(n-1,n),(n-2,n-1),\ldots,(1,2)$ to introduce zeros into $\mathbf{q}_1$. What happens if those same Givens rotations are then applied to I?]

3. Let A be an $m \times n$ real matrix with $m \geq n$. Show that the minimum value of the quotient $\|A\mathbf{x}\|_2/\|\mathbf{x}\|_2$ (taken over all nonzero vectors $\mathbf{x}$) as well as the particular $\mathbf{x}$ that makes this quotient minimum can be determined from the SVD of A.

4. (Exercise 4.24 from Heath.) Let A be an $n \times n$ real rank-one matrix of the form $A = \mathbf{u}\mathbf{v}^T$ where $\mathbf{u}, \mathbf{v}$ are both nonzero and $\mathbf{u}^T\mathbf{v} \neq 0$.

   (a) Show that $\mathbf{u}^T\mathbf{v}$ is an eigenvalue of A. What is its eigenvector?

   (b) What are the other eigenvalues of A? [Hint: normalize $\mathbf{v}$ and extend it to an orthonormal basis.]

   (c) Suppose the power method is applied to A. How many iterations are required for it to converge exactly to the eigenvector corresponding to the dominant eigenvalue? What assumptions must be made about the initial vector?

Appendix: The underdetermined full-rank least-squares problem is defined on p. 272 of Golub and Van Loan and is as follows. Find $\mathbf{x}$ to minimize $\|\mathbf{x}\|_2$ subject to the constraint that $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbf{R}^{m\times n}$ is a given matrix of rank m (hence $m \leq n$) and $\mathbf{b}$ is a given m-vector. An algorithm for this problem is as follows. Factor $A^T = QR$ using Householder reflections. Then the constraint $A\mathbf{x} = \mathbf{b}$ is rewritten $R^TQ^T\mathbf{x} = \mathbf{b}$. Substitute $\mathbf{y} = Q^T\mathbf{x}$, which doesn't change the norm. Hence the new problem is to minimize $\|\mathbf{y}\|_2$ subject to $R^T\mathbf{y} = \mathbf{b}$. Note that $R^T = [L_1, 0]$ where L_1 is an $m \times m$ full rank lower triangular matrix and 0 is a block of $m \times (n-m)$ zeros. We can solve exactly for $\mathbf{y}(1:m)$ using forward substitution on $L_1\mathbf{y}(1:m) = \mathbf{b}$. Then, for minimizing the norm, we take $\mathbf{y}(m+1:n) = \mathbf{0}$. Finally, we take $\mathbf{x} = Q\mathbf{y}$. This $\mathbf{x}$ is computed by applying the Householder transformations computed during the factorization phase to $\mathbf{y}$.