Quiz Solution, CS421

November 12, 1999

1.

(i) T. A is singular, there exist nontrivial x_0 such that $Ax_0 = 0$. If $Ax = b$ has solution, $(x + \alpha x_0)$ is also solution of the equation where α is any number. Thus the quations have either infinitely many solution or no solution.

(ii) F. Consider

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Both $[1\ ,0\ ,0]^T$ and $[1\ ,-1\ ,1]^T$ are solutions.

(iii) T. If $A(i,i) = 0$, then $e_i^T A e_i = A(i,i) = 0$ where e_i is unit vector whose ith component is 1. Thus A is not SPD. Contradiction.

(iv) F. Consider $x = [2\ ,3]$, $y = [1\ ,4]$. Then $\|x\|_2 < \|y\|_2$. But $\|x\|_1 = \|y\|_1$.

(v) T. Let $\tilde{z} = Qz$. Then we consider equation $PAQ^T\tilde{z} = 0$. Let $\tilde{z} = [\tilde{z}_1; \tilde{z}_2]$ where $\tilde{z}_1$ is q vector and $\tilde{z}_2$ is $n-q$ vector. Let $\tilde{z}_2 = 0$. Because $p < q$, there exists nontrivial x such that $Bx = 0$. Thus $z = Q[x\ ,0]^T$ is the solution.

(vi) T. For any n vector v, $v^TAv = v^TB^TBv + \sigma v^Tv = \|Bv\|_2^2 + \sigma\|v\|_2^2$. Because $\sigma > 0$ and $\|Bv\|_2 \geq 0$, $v^TAv > 0$. Thus A is SPD.

(vii) F. Consider $f(x) = x^3$. Let y be the true value of $f(x)$ and $\bar{y}$ be the true value of $f(\bar{x})$. Then $|\bar{y} - y| = |\bar{x}^3 - x^3| \approx |f'(x)||\bar{x} - x| + O(|\bar{x} - x|^2) = 3x^2\mu|x| + O(|\bar{x} - x|^2) \approx 3\mu|y|$

$(viii)$ T. $\|Qv\|_2^2 = v^TQ^TQv = v^Tv = \|v\|_2^2$

2. Consider Taylor expansion

$$f(x+h) = f(x) + f'(x)h + f"(x)h^2/2 + \cdots$$

It gives an estimate of $Mh/2$ for truncation error. Therefore the total error is bounded by

$$G(h) = \frac{2\epsilon|f(x)|}{h} + \frac{Mh}{2}.$$

Take first and second order derivative on G. We have

$$G'(h) = -\frac{2\epsilon|f(x)|}{h^2} + \frac{M}{2}$$

$$G''(h) = \frac{4\epsilon|f(x)|}{h^3}.$$

When $h = 2\sqrt{\epsilon\frac{|f(x)|}{M}}$, $G'(h) = 0$ and $G''(h) > 0$. Therefore G gets its local minimal. That means the bound is minimized when

$$h = 2\sqrt{\epsilon\frac{|f(x)|}{M}}.$$

3(i)
Because B is SPD, $\nabla f(\bar{x})^T d = d^T \nabla f(\bar{x}) = -d^T B d < 0$. This means d is the descent direction.

(ii)

$$B + ww^T = L(I + vv^T)L^T$$

where $Lv = w$. To find v need $O(n^2)$ flops. Using Givens Rotation, there exists orthogonal matrix Q_{n-1} such that

$$Q_{n-1}v = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ * \\ 0 \end{bmatrix}.$$

Similarly, there exists orthogonal matrix Q_{n-2} such that

$$Q_{n-2}Q_{n-1}v = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ 0 \\ 0 \end{bmatrix}.$$

Thus we can find orthogonal matrices $Q_1, \cdots, Q_{n-1}$ such that $Q_1 \cdots Q_{n-1}v = [\|v\|_2, 0, 0 \cdots, 0]^T$. Denote $H = LQ_1 \cdots Q_{n-1}$, then

$$B + ww^T = L(I + vv^T)L^T = LQ_1 \cdots Q_{n-1}(I + E)Q_{n-1}^T \cdots Q_1^T L^T = H(I + E)H^T$$

where

$$E = \begin{bmatrix} \|v\|_2^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Notice $H = LQ_1 \cdots Q_{n-1}$ is a lower Hessenberg matrix. We can compute H in $(n-1)$ for loops. Since Q_i's are Givens rotations, only two columns need to be updated in each step. Therefore the computation of H needs $O(n^2)$ flops. Let $D = diag(\ [\sqrt{1 + \|v\|_2^2}, 1, \cdots, 1]\ )$. Then $D^2 = (I+E)$. The computation of D needs $O(n)$ flops. Denote $\tilde{H} = HD$ (Computation of $\tilde{H}$ needs $O(n^2)$ flops). Then

$$B + ww^T = \tilde{H}\tilde{H}^T$$

Because $\tilde{H}$ is also a lower Hessenberg matrix, by Givens rotation we can find orthogonal matrix $\tilde{Q}_1, \cdots, \tilde{Q}_{n-1}$ such that $L_+ = \tilde{H}\tilde{Q}_1 \cdots Q_{n-1}$ is a lower triangular matrix. This operation also needs $O(n^2)$ flops.