CS421 Problem Set 6 Due: Fri Dec 3/99

1. Let $F(x)$ be a nonlinear vector-valued function with an m-by-n sparse Jacobian matrix $J(x)$. Let W, V be m-by-p and n-by-q known binary matrices, respectively, where p and q are integers. At a given point x assume that the products JV and $W^T J$ have both been computed. Assume the sparsity pattern of J is known. In principle how can it be determined if the nonzeroes of J can be directly recovered (i.e., no arithmetic) from this information? Assuming we allow substitutions (but not matrix factorizations) how can it be determined if the nonzeroes of J are recoverable from this given information?

2. A Jordon block, J_λ, is a square upper bidiagonal matrix with all diagonal entries equal to a constant λ and with the super-diagonal elements equal to unity. For example the Jordan block of order 4 is:

$$J_\lambda = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

Prove that all eigenvectors of J_λ are multiples of e_1.

3. Suppose that matrix A has n distinct eigenvalues $\lambda_1, ..., \lambda_n$. Argue that, with the appropriate shifts μ_k, shifted inverse iteration can generate eigenvectors corresponding to each of the eigenvalues λ_k, $k = 1, ..., n$.

4. Let $x_1, ..., x_k$ be k eigenvectors of A. Suppose Q is an orthogonal matrix, $Q = (Q_1, Q_2)$, where $< Q_1 > = < x_1, ..., x_k >$. Show that $Q_2^T A Q_1 = 0$.

5. Assume A has eigenvectors $x_1, ..., x_n$. Let Q be an orthogonal matrix such that

$$< q_1, ..., q_k > = < x_1, ..., x_k >, \quad k = 1 : n. \tag{1}$$

Show that $Q^T A Q$ is upper triangular.

6. Let $B \in \Re_{m \times n}$ and suppose $rank(B) = n$. Let

$$B = (b_1, ..., b_n) = Q \begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{2}$$

where Q is an orthogonal matrix and R is m-by-m and upper triangular. Prove that

$$< q_1, ..., q_k > = < b_1, ..., b_k >, \;\; k = 1 : n$$

where $Q = (q_1, ..., q_n)$.

7. The QR-method for eigenvalues can be simply expressed: Assign $T_1 \leftarrow A$, and for $k = 1, ...,$

- Factor: $T_k = Q_k R_k$
- Multiply: $T_{k+1} \leftarrow R_k Q_k$

Explain what this does and why it works.

8. Suppose A is a square upper Hessenberg matrix. Show how to define an orthogonal matrix Q such that $R \cdot Q$ is upper-Hessenberg, where $A = QR$, R is upper triangular. In the context of determining eigenvalues, why is this important?

9. Suppose A is real and symmetric. Show how to define an orthogonal matrix Q such that $Q^T AQ$ is tridiagonal. In the context of determining eigenvalues, why is this important? This method appears attractive for sparse computations in that if A is sparse then both the original matrix and the result, $Q^T AQ$, are sparse matrices. Explain why this approach is actually **not** generally attractive in the large sparse setting.

10. Let $A \in \Re^{m \times n}$. Show how to compute orthogonal matrices U, V such that $U^T AV$ is upper bidiagonal.