CS 421 – **Problem Set 2**
Due: Wednesday, September 29, 1999

1. (a) Let $x, y \in \Re^n$ and define $f : \Re \to \Re$, $f(\alpha) = \|x - \alpha y\|_2$. Show that f is minimized when $\alpha = \frac{x^T y}{y^T y}$.
   (b) Let A be an m-by-n matrix with $m < n$. Give an argument to support the claim that there exists a nonzero vector x such that $Ax = 0$.

2. Implement in Matlab a column-oriented algorithm to solve $Tx = b$ where T is a nonsingular upper-triangular matrix. Your code should contain only a single "for loop" (i.e., your code should not contain a nested "for loop"). Test you code on a few examples of dimensions $1, 2, 5, 10, 15, 20$. (Use the Matlab commands "triu" and "rand" to generate your examples.)

3. Repeat the above question in the case where T is lower triangular. (Use "tril" to help generate test examples.)

4. Implement in Matlab an LU-factorization algorithm with partial pivoting. Your code should not contain a nested "for loop". Permutation information should be stored in a single n-vector p; do not use an explicit n-by-n matrix P in your code. Test your code on random problems (use "rand") of sizes $1, 2, 5, 10, 15, 20$; compare your answer to the answer obtained using the Matlab "backslash". (Use "norm"to compare answers.) Compute also the relative norm of the residual vector $Ax - b$, i.e., compute $\frac{\|A\bar{x} - b\|}{\|A\| \cdot \|\bar{x}\|}$, where $\bar{x}$ is the computed solution. Comment on any observations you have.

5. The command "hilb(n)" generates a "Hilbert matrix" of order n. Such matrices are badly conditioned (for large enough n). Test your solver on Hilbert matrices of sizes $5, 10, 15$, and 20. Compute the relative norm of the residual vector. Compare these norm values to the relative residual norms obtained after using the Matlab "backslash" solver. Compare the solution obtained by your solver with the solution obtained with the Matlab "backslash" solver. Compute also the condition number of each matrix ("cond(A)"), and comment on all your results.

6. (a) Show that if $A \in \Re^{m*n}$ is a matrix with $rank(A) = n$ then $C = A^T A$ is a symmetric positive definite matrix.

   (b) Implement, in Matlab, a **recursive** Cholesky factorization routine. Test your implementation on matrices of dimensions $5, 10, 15$, and 20: generate "random" symmetric positive definite matrices using (a) above and solve systems of these dimensions. Compare answers, residuals with the Matlab "backslash" solver as above. **Hint:** Consider the border scheme.