Suggestion Answer for Homework 6, CS421

December 8, 1999

1. See attached notes : P13 - P18.

2.
It is easy to show that the eigenvalue of J_λ is λ. Let $x = [x_1, \cdots, x_4]^T$ be the eigenvector. Then we have

$$\lambda x_i + x_{i+1} = \lambda x_i \qquad \text{for} \quad i = 1, 2, 3$$

$\Longrightarrow$

$$x = [x_1, 0, 0, 0]^T$$

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Choose u_k such that

$$\frac{1}{|\lambda_k - u_k|} = \max_{1 \le i \le k} \frac{1}{|\lambda_i - u_k|}$$

4.
Let q_i, $i \le k$ be an arbitrary vector in Q_1. Since $< Q_1 > = < x_1, \cdots, x_k >$, q_i is linear combination of $\{x_1, \cdots, x_k\}$. Denote

$$q_i = \sum_{j=1}^{k} \alpha_j^i x_j.$$

Since $\{x_1, \cdots, x_k\}$ are eigenvectors,

$$Aq_i = \sum_{j=1}^{k} \alpha_j^i \lambda_j x_j.$$

Because $< Q_1 > = < x_1, \cdots, x_k >$, each x_j is linear combination of $\{q_1, \cdots, q_k\}$. Then Aq_i is linear combination of $\{q_1, \cdots, q_k\}$. Because Q is orthogonal,
$\Longrightarrow$

$$Q_2^T A q_i = 0 \qquad \text{for} \quad i = 1, \cdots, k$$

Since i is arbitrary,

$\Longrightarrow$

$$Q_2^T A Q_1 = 0$$

5.
Because $< q_1, \cdots, q_k >=< x_1, \cdots, x_k > \qquad k = 1 : n,$

$$Q = [x_1, \cdots, x_n] * R$$

where R is an upper triangular matrix. Denote $X = [x_1, \cdots, x_n]$ and $\Lambda = diag[\lambda_1, \cdots, \lambda_n]$ where $\lambda_1, \cdots, \lambda_n$ are eigenvalues. Then

$$AQ = AXR = \Lambda XR = XR\Lambda = Q\Lambda$$

$\Longrightarrow$

$$Q^T A Q = \Lambda$$

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We prove by induction argument.
(i) Because R is an upper triangular matrix, it is obvious that $< b_1 >=< q_1 >$.
(ii) Assume that $< q_1, \cdots, q_k >=< b_1, \cdots, b_k >$ for some $k < n$. Because R is an upper triangular matrix, b_{k+1} is linear combination of $\{q_1, \cdots, q_{k+1}\}$. Then $b_{k+1} \in< q_1, \cdots, q_{k+1} >$. By our assumption,

$$< b_1, \cdots, b_{k+1} >\subseteq< q_1, \cdots, q_{k+1} > .$$

Because q_{k+1} is also linear combination of $\{b_{k+1}, q_1, \cdots, q_k\}$, by our assumption, q_{k+1} is linear combination of $\{b_1, \cdots, b_{k+1}\}$. Then

$$< q_1, \cdots, q_{k+1} >\subseteq< b_1, \cdots, b_{k+1}.$$

Thus

$$< b_1, \cdots, b_{k+1} >=< q_1, \cdots, q_{k+1} > .$$

7.
We consider the following algorithm in stead.

$$Z_1 = A \qquad\qquad Z_1 = \tilde{Q}_1 \tilde{R}_1$$

$$Z_2 = A\tilde{Q}_1 \qquad\qquad Z_2 = \tilde{Q}_2 \tilde{R}_2$$

$$\cdots$$

$$Z_n = A\tilde{Q}_{n-1} \qquad\qquad Z_n = \tilde{Q}_n \tilde{R}_n$$

$$Z_{n+1} = A\tilde{Q}_n \qquad\qquad Z_{n+1} = \tilde{Q}_{n+1}\tilde{R}_{n+1}$$

We can verify that $\tilde{R}_k = R_k$, $\tilde{Q}_k = Q_1 \cdots Q_k$, $T_k = \tilde{Q}_{k-1}^T Z_k$.
Assume $|\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|$.
Denote q_k^1 to be the first column of $\tilde{Q}_k$, the we have

$$Aq_k^1 = \alpha_k q_{k+1}^1$$

where α_k is some constant. Thus sequence $\{q_k^1\}$ are same to the vectors generated by power method. So q_k^1 approaches to the eigenvector corresponding to λ_1.
Next, we consider of q_k^2. Because $< q_k^1, q_k^2 >=< q_{k+1}^1, q_{k+1}^2 >$,

$$Aq_k^2 = \beta_k^1 q_{k+1}^1 + \beta_k^2 q_{k+1}^2.$$

We claim β_k^1 converges to 0 as k goes to ∞. If not, assumer $\beta_k^1 \geq \epsilon > 0$ for any k, by the property of power method, q_k^2 will approach to the eigenvector of λ_1. This is impossible because q_k^2 is perpendicular to q_k^1 for any k. Thus q_k^2 approaches to the eigenvector correspondint to λ_2 as k goes to ∞.
Similarly, we can show q_k^j approaches to the the jth eigenvector as k goes to ∞.

8.
By Givens rotation,

$$G_n \cdots G_1 A = R$$

where G_i is the rotation which kill the ith subdiagonal entry. Then $Q = G_1^{-1} \cdots G_n^{-1}$. We claim that RQ is upper Hessenberg matrix. Let us first consider RG_1^{-1}. Because RG_1^{-1} only need update the first two columns of R,

$$RG_1^{-1} = \begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ 0 & 0 & & \\ \vdots & \vdots & R' & \\ 0 & 0 & & \end{bmatrix}$$

where R' is an upper triangular matrix. By similar arguments, we can prove that RQ is an upper Hessenberg matrix.

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Denote H_1 to be Household matrix such that

$$H_1^T * A = \begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix}.$$

Because A is symmetric,

$$H_1^T * A * H_1 = \begin{bmatrix} * & * & 0 & \cdots & 0 \\ * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}.$$

Similarly, we can find $H_2, \cdots, H_{n-1}$ such that $H_{n-1}^T \cdots H_1^T A H_1 \cdots H_{n-1}$ is tridiagonal. Because the eigenvalues of $Q^T AQ$ is same to those of A, instead of searching for eigenvalues of A, we search for eigenvalues of $Q^T AQ$. Since $Q^T AQ$ is sparse, it will save flops in computation. If A is large sparse matrix, this approach is not generally attractive because the work for transforming A to tridiagonal form is dominant. In this case, searching the eigenvalues directly, for example, using iterative method, will be more efficient.

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Define Housedholder matrices U_1, V_1 such that

$$U_1 A = \begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}$$

$$U_1 A V_1 = \begin{bmatrix} * & * & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}.$$

Furthermore, we can define Householder matrices U_2, V_2 such that

$$U_2 U_1 A V_1 V_2 = \begin{bmatrix} * & * & 0 & 0 & \cdots & 0 \\ 0 & * & * & 0 & \cdots & 0 \\ 0 & 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \vdots & \cdots & * \end{bmatrix}.$$

Similarly we can define Householder matrices $U_3, \cdots, U_n$ and $V_3, \cdots, V_{n-2}$, such that $U_n \cdots U_1 A V_1 \cdots V_{n-2}$ is upper bidiagonal.