CS412/CS413

Introduction to Compilers

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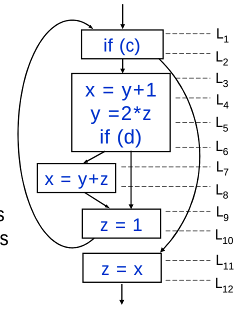
Lecture 26: Dataflow Analysis Frameworks

04 Apr 04

Live Variable Analysis

What are the live variables at each program point?

Method:
1. Define sets of live variables
1. Build constraints
2. Solve constraints



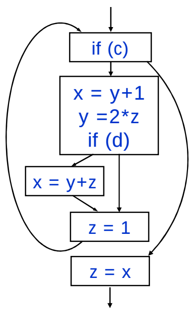
Derive Constraints

Constraints for each instruction:

in[I]=(out[I]-def[I]) ∪ use[I]

Constraints for control flow:

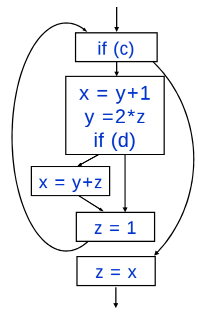
$out[B] = \bigcup_{B' \in succ(B)} in[B']$



L_1
L_2
L_3
L_4
L_5
L_6
L_7
L_8
L_9
L_{10}
L_{11}
L_{12}

Derive Constraints

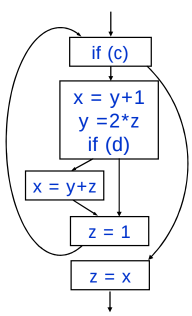
$L_1 = L_2 \cup \{c\}$
$L_2 = L_3 \cup L_{11}$
$L_3 = (L_4 - \{x\}) \cup \{y\}$
$L_4 = (L_5 - \{y\}) \cup \{z\}$
$L_5 = L_6 \cup \{d\}$
$L_6 = L_7 \cup L_9$
$L_7 = (L_8 - \{x\}) \cup \{y,z\}$
$L_8 = L_9$
$L_9 = L_{10} - \{z\}$
$L_{10} = L_1$
$L_{11} = (L_{12} - \{z\}) \cup \{x\}$



L_1
L_2
L_3
L_4
L_5
L_6
L_7
L_8
L_9
L_{10}
L_{11}
L_{12}

Initialization

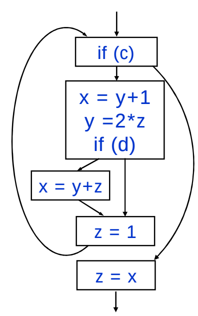
$L_1 = L_2 \cup \{c\}$
$L_2 = L_3 \cup L_{11}$
$L_3 = (L_4 - \{x\}) \cup \{y\}$
$L_4 = (L_5 - \{y\}) \cup \{z\}$
$L_5 = L_6 \cup \{d\}$
$L_6 = L_7 \cup L_9$
$L_7 = (L_8 - \{x\}) \cup \{y,z\}$
$L_8 = L_9$
$L_9 = L_{10} - \{z\}$
$L_{10} = L_1$
$L_{11} = (L_{12} - \{z\}) \cup \{x\}$



$L_1 = \{\}$
$L_2 = \{\}$
$L_3 = \{\}$
$L_4 = \{\}$
$L_5 = \{\}$
$L_6 = \{\}$
$L_7 = \{\}$
$L_8 = \{\}$
$L_9 = \{\}$
$L_{10} = \{\}$
$L_{11} = \{\}$
$L_{12} = \{\}$

Iteration 1

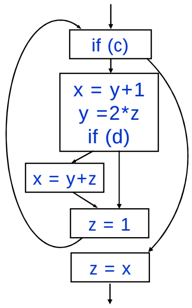
$L_1 = L_2 \cup \{c\}$
$L_2 = L_3 \cup L_{11}$
$L_3 = (L_4 - \{x\}) \cup \{y\}$
$L_4 = (L_5 - \{y\}) \cup \{z\}$
$L_5 = L_6 \cup \{d\}$
$L_6 = L_7 \cup L_9$
$L_7 = (L_8 - \{x\}) \cup \{y,z\}$
$L_8 = L_9$
$L_9 = L_{10} - \{z\}$
$L_{10} = L_1$
$L_{11} = (L_{12} - \{z\}) \cup \{x\}$



$L_1 = \{x,y,z,c,d\}$
$L_2 = \{x,y,z,d\}$
$L_3 = \{y,z,d\}$
$L_4 = \{z,d\}$
$L_5 = \{y,z,d\}$
$L_6 = \{y,z\}$
$L_7 = \{y,z\}$
$L_8 = \{\}$
$L_9 = \{\}$
$L_{10} = \{\}$
$L_{11} = \{x\}$
$L_{12} = \{\}$

Iteration 2

$L_1 = L_2 \cup \{c\}$
$L_2 = L_3 \cup L_{11}$
$L_3 = (L_4 - \{x\}) \cup \{y\}$
$L_4 = (L_5 - \{y\}) \cup \{z\}$
$L_5 = L_6 \cup \{d\}$
$L_6 = L_7 \cup L_9$
$L_7 = (L_8 - \{x\}) \cup \{y,z\}$
$L_8 = L_9$
$L_9 = L_{10} - \{z\}$
$L_{10} = L_1$
$L_{11} = (L_{12} - \{z\}) \cup \{x\}$



$L_1 = \{x,y,z,c,d\}$
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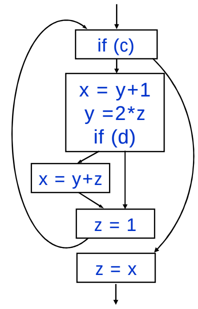
$L_7 = \{y,z,c,d\}$
$L_8 = \{x,y,c,d\}$
$L_9 = \{x,y,c,d\}$
$L_{10} = \{x,y,z,c,d\}$
$L_{11} = \{x\}$
$L_{12} = \{\}$

Fixed-point!

$L_1 = L_2 \cup \{c\}$
$L_2 = L_3 \cup L_{11}$
$L_3 = (L_4 - \{x\}) \cup \{y\}$
$L_4 = (L_5 - \{y\}) \cup \{z\}$
$L_5 = L_6 \cup \{d\}$
$L_6 = L_7 \cup L_9$
$L_7 = (L_8 - \{x\}) \cup \{y,z\}$
$L_8 = L_9$
$L_9 = L_{10} - \{z\}$
$L_{10} = L_1$
$L_{11} = (L_{12} - \{z\}) \cup \{x\}$

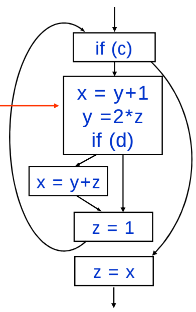


$L_1 = \{x,y,z,c,d\}$
$L_2 = \{x,y,z,c,d\}$
$L_3 = \{y,z,c,d\}$
$L_4 = \{x,z,c,d\}$
$L_5 = \{x,y,z,c,d\}$
$L_6 = \{x,y,z,c,d\}$
$L_7 = \{y,z,c,d\}$
$L_8 = \{x,y,c,d\}$
$L_9 = \{x,y,c,d\}$
$L_{10} = \{x,y,z,c,d\}$
$L_{11} = \{x\}$
$L_{12} = \{\}$

Final Result

x live here !

Final result: sets of live variables at each program point



$L_1 = \{x,y,z,c,d\}$
$L_2 = \{x,y,z,c,d\}$
$L_3 = \{y,z,c,d\}$
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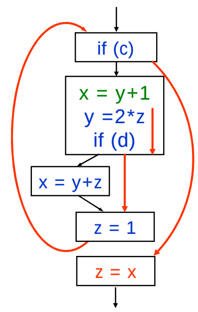
$L_9 = \{x,y,c,d\}$
$L_{10} = \{x,y,z,c,d\}$
$L_{11} = \{x\}$
$L_{12} = \{\}$

Characterize All Executions

The analysis detects that there is an execution that uses the value x = y+1



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$L_{12} = \{\}$

Generalization

- Live variable analysis and detection of available copies are similar:
  - Define some information that they need to compute
  - Build constraints for the information
  - Solve constraints iteratively:
    - The information always "increases" during iteration
    - Eventually, it reaches a fixed point.
- We would like a general framework
  - Framework applicable to many other analyses
  - Live variable/copy propagation = instances of the framework

Dataflow Analysis Framework

- Dataflow analysis = a common framework for many compiler analyses
  - Computes some information at each program point
  - The computed information characterizes all possible executions of the program
- Basic methodology:
  - Describe information about the program using an algebraic structure called a lattice
  - Build constraints that show how instructions and control flow influence the information in terms of values in the lattice
  - Iteratively solve constraints

Partial Order Relations

- Lattice definition builds on the concept of a partial order relation
- A partial order $(P, \sqsubseteq)$ consists of:
  - A set P
  - A partial order relation $\sqsubseteq$ that is:
    1. Reflexive $\quad x \sqsubseteq x$
    2. Anti-symmetric $\quad x \sqsubseteq y,\ y \sqsubseteq x \Rightarrow x = y$
    3. Transitive: $\quad x \sqsubseteq y,\ y \sqsubseteq z \Rightarrow x \sqsubseteq z$
- Called a "*partial* order" because not all elements are comparable, in contrast with a *total* order, in which
  - $\neg$4. Total $\quad x \sqsubseteq y$ or $y \sqsubseteq x$

Example

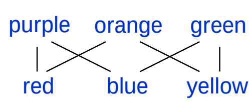
- P is {red, blue, yellow, purple, orange, green}
- $\sqsubseteq$

  red $\sqsubseteq$ purple, red $\sqsubseteq$ orange,
  blue $\sqsubseteq$ purple, blue $\sqsubseteq$ green,
  yellow $\sqsubseteq$ orange, blue $\sqsubseteq$ green,
  red $\sqsubseteq$ red,
  blue $\sqsubseteq$ blue,
  yellow $\sqsubseteq$ yellow,
  purple $\sqsubseteq$ purple,
  orange $\sqsubseteq$ orange,
  green $\sqsubseteq$ green

Hasse Diagrams

- Hasse diagram = graphical representation of a partial order, where x is below y when $x \sqsubseteq y$ and $x \neq y$

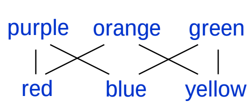
x and y on same line implies they are incomparable



Lattices and Lower/Upper Bounds

- Lattice definition uses the concept of lower and upper bounds
- If $(P, \sqsubseteq)$ is a partial order and $S \subseteq P$, then:
  1. $x \in P$ is a lower bound of S if $x \sqsubseteq y$, for all $y \in S$
  2. $x \in P$ is an upper bound of S if $y \sqsubseteq x$, for all $y \in S$
- There may be multiple lower and upper bounds of the same set S

Example, cont.



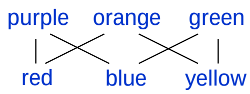
red is lower bound for {purple, orange}
blue is lower bound for {purple, green}
yellow is lower bound for {orange, green}
no lower bound for {purple, orange, green}
no lower bound for {red, blue}
no lower bound for {red, yellow}
no lower bound for {blue, yellow},
etc.

purple is upper bound for {red, blue}
orange is upper bound for {red, yellow}
green is upper bound for {orange, green}
no upper bound for {red, bule, yellow}
no upper bound for {purple, orange}
no upper bound for {orange, green}
no upper bound for {purple, green}
etc.

LUB and GLB

- Define least upper bounds (LUB) and greatest lower bounds (GLB)
- If $(P, \sqsubseteq)$ is a partial order and $S \subseteq P$, then:
  1. $x \in P$ is GLB of S if:
     a) x is a lower bound of S
     b) $y \sqsubseteq x$, for any lower bound y of S
  2. $x \in P$ is a LUB of S if:
     a) x is an upper bound of S
     b) $x \sqsubseteq y$, for any upper bound y of S
- ... are GLB and LUB unique?

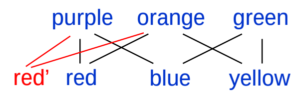
Example, cont.



red is GLB for {purple, orange}
blue is GLB for {purple, green}
yellow is GLB for {orange, green}

purple is LUB for {red, blue}
orange is LUB for {red, yellow}
green is LUB for {orange, green}

Example’



blue is GLB for {purple, green}
yellow is GLB for {orange, green}

red’ is a lower bound for {purple, orange}
red is a lower bound for {purple, orange}
There is no GLB for {purple, orange}

purple is LUB for {red, blue}
orange is LUB for {red, yellow}
green is LUB for {orange, green}
purple is LUB for {red’, blue}
orange is LUB for {red’, yellow}

Example”

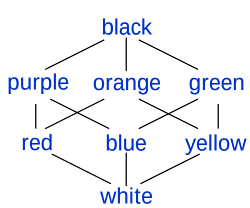
- P is natural numbers {0, 1, 2, 3, … }
- ⊑ is ≤

Every finite subset of P has a GLB and LUB
No infinite subset of P has a LUB

Lattices

- A pair (L,⊑) is a lattice if:
  1. (L,⊑) is a partial order
  2. Any finite subset S ⊆ L has a LUB and a GLB
- Can define two operators in lattices:
  1. Meet operator: x ⊓ y = GLB({x,y})
  2. Join operator: x ⊔ y = LUB({x,y})
- Meet and join are well-defined for lattices

Example



white is GLB for {red, blue, yellow}

black is LUB for {purple, orange, green}

Complete Lattices

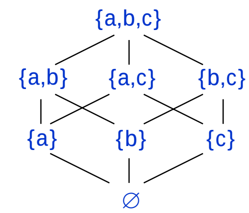
- A pair (L,⊑) is a complete lattice if:
  1. (L,⊑) is a partial order
  2. Any subset S ⊆ L has a LUB and a GLB
- Can define meet and join operators
- Can also define two special elements:
  1. Bottom element: ⊥ = GLB(L)
  2. Top element: ⊤ = LUB(L)
- All finite lattices are complete

Example Lattice

- Consider S = {a,b,c} and its power set P = {∅, {a}, {b}, {c}, {a,b}, {b,c}, {a,c} {a,b,c}}
- Define partial order as set inclusion: $X \subseteq Y$
  - Reflexive $X \subseteq X$
  - Anti-symmetric $X \subseteq Y,\ Y \subseteq X \Rightarrow X = Y$
  - Transitive $X \subseteq Y,\ Y \subseteq Z \Rightarrow X \subseteq Z$
- Also, for any two elements of P, there is a set that includes both and another set that is included in both
- Therefore $(P, \subseteq)$ is a (complete) lattice

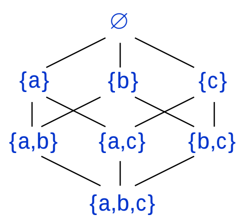
Power Set Lattice

- Partial order: ⊆ (set inclusion)
- Meet: ∩ (set intersection)
- Join: ∪ (set union)
- Top element: {a,b,c} (whole set)
- Bottom element: ∅ (empty set)



Reversed Lattice

- Partial order: ⊇ (set inclusion)
- Meet: ∪ (set union)
- Join: ∩ (set intersection)
- Top element: ∅ (empty set)
- Bottom element: {a,b,c} (whole set)



Relation To Dataflow Analysis

- Information computed by live variable analysis and available copies can be expressed as elements of lattices
- Live variables: if V is the set of all variables in the program and P the power set of V, then:
  - (P,⊆) is a lattice
  - sets of live variables are elements of this lattice

Relation To Analysis of Programs

- Copy Propagation:
  - V is the set of all variables in the program
  - V x V the cartesian product representing all possible copy instructions
  - P the power set of V x V
- Then:
  - (P,⊆) is a lattice
  - sets of available copies are lattice elements

More About Lattices

- In a lattice $(L, \sqsubseteq)$, the following are equivalent:
  1. $x \sqsubseteq y$
  2. $x \sqcap y = x$
  3. $x \sqcup y = y$
- Note: meet and join operations were defined using the partial order relation

Proof

- Prove that $x \sqsubseteq y$ implies $x \sqcap y = x$:
  - x is a lower bound of {x,y}
  - All lower bounds of {x,y} are less than x,y
  - In particular, they are less than x
- Prove that $x \sqcap y = x$ implies $x \sqsubseteq y$:
  - x is a lower bound of {x,y}
  - x is less than x and y
  - In particular, x is less than y

Proof

- Prove that $x \sqsubseteq y$ implies $x \sqcup y = y$:
  - y is an upper bound of {x,y}
  - All upper bounds of {x,y} greater than x,y
  - In particular, they are greater than y
- Prove that $x \sqcup y = y$ implies $x \sqsubseteq y$:
  - y is a upper bound of {x,y}
  - y is greater than x and y
  - In particular, y is greater than x

Properties of Meet and Join

- The meet and join operators are:
  1. Associative $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
  2. Commutative $x \sqcap y = y \sqcap x$
  3. Idempotent: $x \sqcap x = x$
- Property: If "$\sqcap$" is an associative, commutative, and idempotent operator, then the relation "$\sqsubseteq$" defined as $x \sqsubseteq y$ iff $x \sqcap y = x$ is a partial order
- Above property provides an alternative definition of a partial orders and lattices starting from the meet (join) operator

Using Lattices

- Assume information we want to compute in a program is expressed using a lattice L
- To compute the information at each program point we need to:
  - Determine how each instruction in the program changes the information
  - Determine how information changes at join/split points in the control flow

Transfer Functions

- Dataflow analysis defines a transfer function $F : L \to L$ for each instruction in the program
- Describes how the instruction modifies the information
- Consider in[I] is information before I, and out[I] is information after I
- Forward analysis: out[I] = F(in[I])
- Backward analysis: in[I] = F(out[I])

Basic Blocks

- Can extend the concept of transfer function to basic blocks using function composition
- Consider:
  - Basic block B consists of instructions $(I_1, \ldots, I_n)$ with transfer functions $F_1, \ldots, F_n$
  - in[B] is information before B
  - out[B] is information after B
- Forward analysis:
  $out[B] = F_n(\ldots(F_1(in[B]))) = F_n \circ \ldots \circ F_1(in[B])$
- Backward analysis:
  $in[I] = F_1(\ldots (F_n(out[i]))) = F_1 \circ \ldots \circ F_n(out[B])$

Split/Join Points

- Dataflow analysis uses meet/join operations at split/join points in the control flow
- Consider in[B] is lattice information at beginning of block B and out[B] is lattice information at end of B
- Forward analysis: $in[B] = \sqcap \{out[B'] \mid B' \in pred(B)\}$
- Backward analysis: $out[B] = \sqcap \{in[B'] \mid B' \in succ(B)\}$
- Can alternatively use join operation $\sqcup$ (equivalent to using the meet operation $\sqcap$ in the reversed lattice)