

CS 4110

Programming Languages & Logics

Lecture 32

Normalization



Normalization

The simply-typed lambda calculus enjoys a remarkable property:

Every well-typed program terminates.

Simply-Typed Lambda Calculus

Syntax

expressions

$e ::= x \mid \lambda x:\tau. e \mid e_1 e_2 \mid ()$

values

$v ::= \lambda x:\tau. e \mid ()$

types

$\tau ::= \mathbf{unit} \mid \tau_1 \rightarrow \tau_2$

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Dynamic Semantics

$E ::= [\cdot] \mid E e \mid v E$

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x:\tau. e) v \rightarrow e\{v/x\}}$$

Simply-Typed Lambda Calculus

Static Semantics

$$\frac{}{\Gamma \vdash () : \mathbf{unit}} \text{ T-UNIT}$$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{ T-VAR}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau. e : \tau \rightarrow \tau'} \text{ T-ABS}$$

$$\frac{\Gamma \vdash e_1 : \tau \rightarrow \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'} \text{ T-APP}$$

Supporting Lemmas

Lemma (Inversion)

- *If $\Gamma \vdash x:\tau$ then $\Gamma(x) = \tau$*
- *If $\Gamma \vdash \lambda x:\tau_1. e:\tau$ then $\tau = \tau_1 \rightarrow \tau_2$ and $\Gamma, x:\tau_1 \vdash e:\tau_2$.*
- *If $\Gamma \vdash e_1 e_2:\tau$ then $\Gamma \vdash e_1:\tau' \rightarrow \tau$ and $\Gamma \vdash e_2:\tau'$.*

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Lemma (Canonical Forms)

- If $\Gamma \vdash v:\mathbf{unit}$ then $v = ()$
- If $\Gamma \vdash v:\tau_1 \rightarrow \tau_2$ then $v = \lambda x:\tau_1. e$ and $\Gamma, x:\tau_1 \vdash e:\tau_2$.

First Attempt

Theorem (Normalization)

If $\vdash e : \tau$ then there exists a value v such that $e \rightarrow^ v$.*

Logical Relations

Idea: define a set with the following properties:

- At base types the set contains all expressions satisfying some property.
- At function types, the set contains all expressions such that the property is preserved whenever we apply the function to an argument of appropriate type that is also in the set.

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In our setting, the property will concern normalization...

Logical Relation

Definition (Logical Relation)

- $R_{\text{unit}}(e)$ iff $\vdash e : \text{unit}$ and e halts.
- $R_{\tau_1 \rightarrow \tau_2}(e)$ iff $\vdash e : \tau_1 \rightarrow \tau_2$ and e halts, and for every e' such that $R_{\tau_1}(e')$ we have $R_{\tau_2}(e\ e')$.

Supporting Lemmas

Lemma

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Lemma (Goal)

If $\vdash e:\tau$ then $R_\tau(e)$

Main Lemma

Lemma (Goal – Strengthened)

If

- $x_1:\tau_1, \dots, x_k:\tau_k \vdash e:\tau,$
- v_1 through v_k are values such that $\vdash v_1:\tau_1$ through $\vdash v_k:\tau_k$, and
- $R_{\tau_1}(v_1)$ through $R_{\tau_k}(v_k),$

then $R_\tau(e\{v_1/x_1\} \dots \{v_k/x_k\})$.