CS 4110
Programming Languages & Logics

Lecture 23
De Bruijn Notation and Combinators

de Bruijn Notation

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Abstractions have lost their variables!

Variables are replaced with numerical indices!

Examples

Here are some terms written in standard and de Bruijn notation:

Standard	**de Bruijn**
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$\lambda z.\ z$	

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$\lambda z.\ z$	$\lambda.\ 0$
$\lambda x.\ \lambda y.\ x$	$\lambda.\ \lambda.\ 1$
$\lambda x.\ \lambda y.\ \lambda s.\ \lambda z.\ x\ s\ (y\ s\ z)$	

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$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$	$(\lambda.\ 0\ 0)\ (\lambda.\ 0\ 0)$
$(\lambda x.\ \lambda x.\ x)\ (\lambda y.\ y)$	$(\lambda.\ \lambda.\ 0)\ (\lambda.\ 0)$

Free variables

To represent a λ-expression that contains free variables in de Bruijn notation, we need a way to map the free variables to integers.

We will work with respect to a map Γ from variables to integers called a *context*.

Examples:

Suppose that Γ maps x to 0 and y to 1.

- Representation of $x\ y$ is 0 1
- Representation of $\lambda z.\ x\ y\ z\ \lambda.$ 1 2 0

Shifting

To define substitution, we will need an operation that shifts by i the variables above a cutoff c:

$$\begin{array}{rcl} \uparrow_c^i (n) & = & \begin{cases} n & \text{if } n < c \\ n + i & \text{otherwise} \end{cases} \\ \uparrow_c^i (\lambda.e) & = & \lambda.(\uparrow_{c+1}^i e) \\ \uparrow_c^i (e_1\ e_2) & = & (\uparrow_c^i e_1)\,(\uparrow_c^i e_2) \end{array}$$

The cutoff c keeps track of the variables that were bound in the original expression and so should not be shifted.

The cutoff is 0 initially.

Substitution

Now we can define substitution:

$$
\begin{array}{rcl}
n\{e/m\} & = & \begin{cases} e & \text{if } n = m \\ n & \text{otherwise} \end{cases} \\
(\lambda.e_1)\{e/m\} & = & \lambda.e_1\{(\uparrow_0^1 e)/m+1\} \\
(e_1\ e_2)\{e/m\} & = & (e_1\{e/m\})\ (e_2\{e/m\})
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The β rule for terms in de Bruijn notation is just:

$$\frac{}{(\lambda.e_1)\ e_2\ \rightarrow \uparrow_0^{-1} (e_1\{\uparrow_0^1 e_2/0\})}\ \beta$$

Example

Consider the term $(\lambda u.\lambda v.u\ x)\ y$ with respect to a context where $\Gamma(x) = 0$ and $\Gamma(y) = 1$.

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\rightarrow\ & \uparrow_0^{-1} ((\lambda.1\ 2)\{(\uparrow_0^1 1)/0\}) \\
=\ & \uparrow_0^{-1} ((\lambda.1\ 2)\{2/0\}) \\
=\ & \uparrow_0^{-1} \lambda.((1\ 2)\{(\uparrow_0^1 2)/(0+1)\}) \\
=\ & \uparrow_0^{-1} \lambda.((1\ 2)\{3/1\}) \\
=\ & \uparrow_0^{-1} \lambda.(1\{3/1\})\ (2\{3/1\}) \\
=\ & \uparrow_0^{-1} \lambda.3\ 2
\end{aligned}
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= & \uparrow_0^{-1} \lambda.3\ 2 \\
= & \lambda.2\ 1
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\end{array}
$$

which, in standard notation (with respect to Γ), is the same as $\lambda v.y\ x$.

Combinators

Another way to avoid the issues having to do with free and bound variable names in the λ-calculus is to work with closed expressions or *combinators*.

With just three combinators, we can encode the entire λ-calculus.

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$$\mathsf{K} = \lambda x.\lambda y.\, x$$
$$\mathsf{S} = \lambda x.\lambda y.\lambda z.\, x\, z\, (y\, z)$$
$$\mathsf{I} = \lambda x.\, x$$

Combinators

We can even define independent evaluation rules that don't depend on the λ-calculus at all.

Behold the "SKI-calculus":

$$\begin{aligned} &\mathsf{K}\ e_1\ e_2 \to e_1 \\ &\mathsf{S}\ e_1\ e_2\ e_3 \to e_1\ e_3\ (e_2\ e_3) \\ &\mathsf{I}\ e \to e \end{aligned}$$

You would never want to program in this language—it doesn't even have variables!—but it's exactly as powerful as the λ-calculus.

Bracket Abstraction

The function $[x]$ that takes a combinator term M and builds another term that behaves like $\lambda x.M$:

$$\begin{array}{rcll} [x]\, x & = & \mathsf{I} & \\ [x]\, N & = & \mathsf{K}\, N & \text{where } x \notin fv(N) \\ [x]\, N_1\, N_2 & = & \mathsf{S}\, ([x]\, N_1)\, ([x]\, N_2) & \end{array}$$

The idea is that $([x]\, M)\, N \to M\{N/x\}$ for every term N.

Bracket Abstraction

We then define a function $(e)*$ that maps a λ-calculus expression to a combinator term:

$$\begin{aligned} (x)* &= x \\ (e_1\ e_2)* &= (e_1)*\ (e_2)* \\ (\lambda x.e)* &= [x]\ (e)* \end{aligned}$$

Example

As an example, the expression $\lambda x.\lambda y.\ x$ is translated as follows:

$$
\begin{aligned}
& (\lambda x.\lambda y.\ x)* \\
= \ & [x]\ (\lambda y.\ x)* \\
= \ & [x]\ ([y]\ x) \\
= \ & [x]\ (\mathsf{K}\ x) \\
= \ & (\mathsf{S}\ ([x]\ \mathsf{K})\ ([x]\ x)) \\
= \ & \mathsf{S}\ (\mathsf{K}\ \mathsf{K})\ \mathsf{I}
\end{aligned}
$$

No variables in the final combinator term!

Example

We can check that this behaves the same as our original λ-expression by seeing how it evaluates when applied to arbitrary expressions e_1 and e_2.

$$
\begin{array}{ll}
 & (\lambda x.\lambda y.\, x)\; e_1\; e_2 \\
\rightarrow & (\lambda y.\, e_1)\; e_2 \\
\rightarrow & e_1
\end{array}
$$

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\to & (\lambda y.\, e_1)\; e_2 \\
\to & e_1
\end{array}
$$

and

$$
\begin{array}{ll}
 & (\mathsf{S}\;(\mathsf{K}\;\mathsf{K})\;\mathsf{I})\; e_1\; e_2 \\
\to & (\mathsf{K}\;\mathsf{K}\;e_1)\;(\mathsf{I}\;e_1)\; e_2 \\
\to & \mathsf{K}\;e_1\; e_2 \\
\to & e_1
\end{array}
$$

SKI Without I

Looking back at our definitions...

$$\mathsf{K}\ e_1\ e_2 \to e_1$$
$$\mathsf{S}\ e_1\ e_2\ e_3 \to e_1\ e_3\ (e_2\ e_3)$$
$$\mathsf{I}\ e \to e$$

...*I* isn't strictly necessary. It behaves the same as S K K.

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$$\begin{aligned} &\mathsf{K}\ e_1\ e_2 \to e_1 \\ &\mathsf{S}\ e_1\ e_2\ e_3 \to e_1\ e_3\ (e_2\ e_3) \\ &\mathsf{I}\ e \to e \end{aligned}$$

...*I* isn't strictly necessary. It behaves the same as S K K.

Our example becomes:

$$\mathsf{S}\ (\mathsf{K}\ \mathsf{K})\ (\mathsf{S}\ \mathsf{K}\ \mathsf{K})$$

SKI Without SKI?

You can go one step farther!

$$\begin{aligned}\iota &\triangleq \lambda f.\, f\, S\, K \\ &= \lambda f.\, f\, (\lambda a.\, \lambda b.\, \lambda c.\, ((a\ c)\ (b\ c)))\ \lambda d.\, \lambda e.\, d\end{aligned}$$

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So:

$$\begin{aligned}S &\equiv_\beta \iota\, (\iota\, (\iota\, (\iota\, \iota))) \\ K &\equiv_\beta \iota\, (\iota\, (\iota\, \iota)) \\ I &\equiv_\beta \iota\, \iota\end{aligned}$$

A single combinator suffices!