CS 4110

Programming Languages & Logics

Lecture 18
Evaluation Contexts and Definitional Translation

Review: Call-by-Value

Here are the syntax and CBV semantics of λ-calculus:

$$e ::= x \mid \lambda x.\, e \mid e_1\, e_2$$
$$v ::= \lambda x.\, e$$

$$\frac{e_1 \to e_1'}{e_1\, e_2 \to e_1'\, e_2} \qquad \frac{e \to e'}{v\, e \to v\, e'}$$

$$\frac{}{(\lambda x.\, e)\, v \to e\{v/x\}}\ \beta$$

There are two kinds of rules: *congruence rules* that specify evaluation order and *computation rules* that specify the "interesting" reductions.

Evaluation Contexts

Evaluation contexts let us separate out these two kinds of rules.

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An evaluation context E is an expression with a "hole" in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

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An evaluation context E is an expression with a "hole" in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

$$E ::= [\cdot] \mid E\, e \mid v\, E$$

We write $E[e]$ to mean the evaluation context E where the hole has been replaced with the expression e.

Examples

$$
\begin{aligned}
E_1 &= [\cdot]\,(\lambda x.\, x) \\
E_1[\lambda y.\, y\, y] &= (\lambda y.\, y\, y)\, \lambda x.\, x
\end{aligned}
$$

Examples

$$
\begin{aligned}
E_1 &= [\cdot]\,(\lambda x.\, x) \\
E_1[\lambda y.\, y\, y] &= (\lambda y.\, y\, y)\; \lambda x.\, x
\end{aligned}
$$

$$
\begin{aligned}
E_2 &= (\lambda z.\, z\, z)\,[\cdot] \\
E_2[\lambda x.\, \lambda y.\, x] &= (\lambda z.\, z\, z)\,(\lambda x.\, \lambda y.\, x)
\end{aligned}
$$

Examples

$$
\begin{aligned}
E_1 &= [\cdot]\,(\lambda x.\, x) \\
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\begin{aligned}
E_2 &= (\lambda z.\, z\, z)\, [\cdot] \\
E_2[\lambda x.\, \lambda y.\, x] &= (\lambda z.\, z\, z)\, (\lambda x.\, \lambda y.\, x)
\end{aligned}
$$

$$
\begin{aligned}
E_3 &= ([\cdot]\, \lambda x.\, x\, x)\, ((\lambda y.\, y)\, (\lambda y.\, y)) \\
E_3[\lambda f.\, \lambda g.\, f\, g] &= ((\lambda f.\, \lambda g.\, f\, g)\, \lambda x.\, x\, x)\, ((\lambda y.\, y)\, (\lambda y.\, y))
\end{aligned}
$$

CBV With Evaluation Contexts

With evaluation contexts, we can define the evaluation semantics for the CBV λ-calculus with just two rules: one for evaluation contexts, and one for β-reduction.

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With this syntax:

$$E ::= [\cdot] \mid E\, e \mid v\, E$$

The small-step rules are:

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x.\, e)\, v \to e\{v/x\}}\ \beta$$

CBN With Evaluation Contexts

We can also define the semantics of CBN λ-calculus with evaluation contexts.

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For call-by-name, the syntax for evaluation contexts is different:

$$E ::= [\cdot] \mid E\, e$$

But the small-step rules are the same:

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x.\, e)\, e' \rightarrow e\{e'/x\}}\ \beta$$

Definitional Translation

We know how to encode Booleans, conditionals, natural numbers, and recursion in λ-calculus.

Can we define a *real* programming language by translating everything in it into the λ-calculus?

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Can we define a *real* programming language by translating everything in it into the λ-calculus?

In definitional translation, we define a denotational semantics where the target is a simpler programming language instead of mathematical objects.

Multi-Argument λ-calculus

Let's define a version of the λ-calculus that allows functions to take multiple arguments.

$$e ::= x \mid \lambda x_1, \ldots, x_n.\, e \mid e_0\, e_1\ \ldots\ e_n$$

Multi-Argument λ-calculus

We can define a CBV operational semantics:

$$E ::= [\cdot] \mid v_0 \ \ldots \ v_{i-1} \, E \, e_{i+1} \ \ldots \ e_n$$

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x_1, \ldots, x_n .\, e_0) \, v_1 \ \ldots \ v_n \to e_0\{v_1/x_1\}\{v_2/x_2\} \ldots \{v_n/x_n\}} \ \beta$$

The evaluation contexts ensure that we evaluate multi-argument applications $e_0 \, e_1 \ \ldots \ e_n$ from left to right.

Definitional Translation

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We can define a translation $\mathcal{T}[\![\cdot]\!]$ that takes an expression in the multi-argument λ-calculus and returns an equivalent expression in the pure λ-calculus.

$$
\begin{aligned}
\mathcal{T}[\![x]\!] &= x \\
\mathcal{T}[\![\lambda x_1, \ldots, x_n.\, e]\!] &= \lambda x_1.\ \ldots\ \lambda x_n.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![e_0\; e_1\; e_2\ \ldots\ e_n]\!] &= (\ldots((\mathcal{T}[\![e_0]\!]\; \mathcal{T}[\![e_1]\!])\; \mathcal{T}[\![e_2]\!])\ldots\ \mathcal{T}[\![e_n]\!])
\end{aligned}
$$

This translation *curries* the multi-argument λ-calculus.

Products (Pairs) and Let

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_1\ e_2 \\
\mid {} & (e_1, e_2) \\
\mid {} & \#1\ e \\
\mid {} & \#2\ e \\
\mid {} & \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2
\end{aligned}
$$

$$
\begin{aligned}
v ::= {} & \lambda x.\, e \\
\mid {} & (v_1, v_2)
\end{aligned}
$$

Products (Pairs) and Let

Evaluation Contexts

$$
\begin{aligned}
E ::= {} & [\cdot] \\
& \mid E\ e \\
& \mid v\ E \\
& \mid (E, e) \\
& \mid (v, E) \\
& \mid \#1\ E \\
& \mid \#2\ E \\
& \mid \text{let}\ x = E\ \text{in}\ e_2
\end{aligned}
$$

Products (Pairs) and Let

Semantics

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x.\, e)\, v \rightarrow e\{v/x\}}\ \beta$$

$$\frac{}{\#1\,(v_1, v_2) \rightarrow v_1} \qquad \frac{}{\#2\,(v_1, v_2) \rightarrow v_2}$$

$$\frac{}{\text{let}\, x = v \,\text{in}\, e \rightarrow e\{v/x\}}$$

Products (Pairs) and Let

Translation

$$
\begin{aligned}
\mathcal{T}[\![x]\!] &= x \\
\mathcal{T}[\![\lambda x.\, e]\!] &= \lambda x.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![e_1\; e_2]\!] &= \mathcal{T}[\![e_1]\!]\; \mathcal{T}[\![e_2]\!] \\
\mathcal{T}[\![(e_1, e_2)]\!] &= (\lambda x.\, \lambda y.\, \lambda f.\, f\, x\, y)\; \mathcal{T}[\![e_1]\!]\; \mathcal{T}[\![e_2]\!] \\
\mathcal{T}[\![\#1\; e]\!] &= \mathcal{T}[\![e]\!]\; (\lambda x.\, \lambda y.\, x) \\
\mathcal{T}[\![\#2\; e]\!] &= \mathcal{T}[\![e]\!]\; (\lambda x.\, \lambda y.\, y) \\
\mathcal{T}[\![\mathsf{let}\; x = e_1\; \mathsf{in}\; e_2]\!] &= (\lambda x.\, \mathcal{T}[\![e_2]\!])\; \mathcal{T}[\![e_1]\!]
\end{aligned}
$$

Laziness

Consider the call-by-name λ-calculus...

Syntax

$$\begin{aligned} e ::= {} & x \\ \mid {} & e_1\ e_2 \\ \mid {} & \lambda x.\, e \end{aligned}$$

$$v ::= \lambda x.\, e$$

Semantics

$$\frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2} \qquad \frac{}{(\lambda x.\, e_1)\ e_2 \to e_1\{e_2/x\}}\ \beta$$

Laziness

Translation

$$
\begin{aligned}
\mathcal{T}[\![x]\!] &= x\,(\lambda y.\, y) \\
\mathcal{T}[\![\lambda x.\, e]\!] &= \lambda x.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![e_1\; e_2]\!] &= \mathcal{T}[\![e_1]\!]\,(\lambda z.\, \mathcal{T}[\![e_2]\!]) \qquad \text{z is not a free variable of e_2}
\end{aligned}
$$

References

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1
\end{aligned}
$$

$$v ::= \lambda x.\, e$$

References

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1 \\
\mid {} & \mathsf{ref}\ e
\end{aligned}
$$

$$v ::= \lambda x.\, e$$

References

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1 \\
\mid {} & \mathsf{ref}\ e \\
\mid {} & !e
\end{aligned}
$$

$$v ::= \lambda x.\, e$$

References

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1 \\
\mid {} & \mathsf{ref}\ e \\
\mid {} & \,!e \\
\mid {} & e_1 := e_2
\end{aligned}
$$

$$
v ::= \lambda x.\, e
$$

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e ::= {} & x \\
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$$v ::= \lambda x.\, e$$

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v ::=\ & \lambda x.\, e \\
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E ::= {} & [\cdot] \\
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\end{aligned}
$$

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\begin{aligned}
E ::= {} & [\cdot] \\
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\mid {} & v\, E \\
\mid {} & \text{ref } E \\
\mid {} & \;!E \\
\mid {} & E := e
\end{aligned}
$$

References

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\begin{aligned}
E ::= {} & [\cdot] \\
\mid {} & E\ e \\
\mid {} & v\ E \\
\mid {} & \mathrm{ref}\ E \\
\mid {} & !E \\
\mid {} & E := e \\
\mid {} & v := E
\end{aligned}
$$

References

Semantics

$$\frac{\langle \sigma, e \rangle \to \langle \sigma', e' \rangle}{\langle \sigma, E[e] \rangle \to \langle \sigma', E[e'] \rangle} \qquad \frac{}{\langle \sigma, (\lambda x.\, e)\; v \rangle \to \langle \sigma, e\{v/x\} \rangle}\;\beta$$

$$\frac{\ell \notin dom(\sigma)}{\langle \sigma, \mathsf{ref}\; v \rangle \to \langle \sigma[\ell \mapsto v], \ell \rangle} \qquad \frac{\sigma(\ell) = v}{\langle \sigma,\; !\ell \rangle \to \langle \sigma, v \rangle}$$

$$\frac{}{\langle \sigma, \ell := v \rangle \to \langle \sigma[\ell \mapsto v], v \rangle}$$

References

Translation

...left as an exercise to the reader. ;-)

Adequacy

How do we know if a translation is correct?

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Every target evaluation should represent a source evaluation...

Definition (Soundness)

$\forall e \in \mathbf{Exp}_{\text{src}}.$ if $\mathcal{T}[\![e]\!] \rightarrow^*_{\text{trg}} v'$ then $\exists v.\ e \rightarrow^*_{\text{src}} v$
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and v' equivalent to v

...and every source evaluation should have a target evaluation:

Definition (Completeness)

$\forall e \in \mathbf{Exp}_{\mathrm{src}}.$ if $e \rightarrow^*_{\mathrm{src}} v$ then $\exists v'.\ \mathcal{T}[\![e]\!] \rightarrow^*_{\mathrm{trg}} v'$
and v' equivalent to v