# CS 4110

# Programming Languages & Logics

## Lecture 15
## Weakest Preconditions

# Generating Preconditions

To fill in a precondition:

$$\{\quad\}\, c\, \{Q\}$$

there are many possible preconditions—and some are more useful than others.

# Weakest Preconditions

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More formally...

## Definition (Weakest Liberal Precondition)

$P$ is a weakest liberal precondition of $c$ and $Q$ written $wlp(c, Q)$ if:

$$\forall \sigma, I.\ \sigma \vDash_I P \iff (\mathcal{C}[\![c]\!]\ \sigma) \text{ undefined } \vee (\mathcal{C}[\![c]\!]\sigma) \vDash_I Q$$

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where

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\begin{aligned}
F_0(P) &= \textbf{true} \\
F_{i+1}(P) &= (\neg b \implies P) \wedge (b \implies wlp(c, F_i(P)))
\end{aligned}
$$

# Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

```
p := getPacket();
processPacket(p);
assert P_safe
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assert P_filter(p);
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$P_{\text{filter}}$ should be the *weakest* precondition to avoid ruling out legitimate inputs.

David Brumley, Hao Wang, Somesh Jha, and Dawn Song. "Creating Vulnerability Signatures Using Weakest Preconditions." In *Computer Security Foundations* (CSF), 2007.

# Properties of Weakest Preconditions

## Lemma (Correctness of Weakest Preconditions)

$$\forall c \in \textbf{Com}, Q \in \textbf{Assn}.$$
$$\vDash \{wlp(c, Q)\}\, c\, \{Q\} \;\; and$$
$$\forall R \in \textbf{Assn}.\ \vDash \{R\}\, c\, \{Q\} \;\; implies \;\; (R \implies wlp(c, Q))$$

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## Lemma (Provability of Weakest Preconditions)

$$\forall c \in \textbf{Com}, Q \in \textbf{Assn}.\ \vdash \{wlp(c, Q)\}\, c\, \{Q\}$$

# Soundness and Completeness

Soundness: If we can prove it, then it's actually true.

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If $\vdash \{P\}\, c\, \{Q\}$ then $\models \{P\}\, c\, \{Q\}$.

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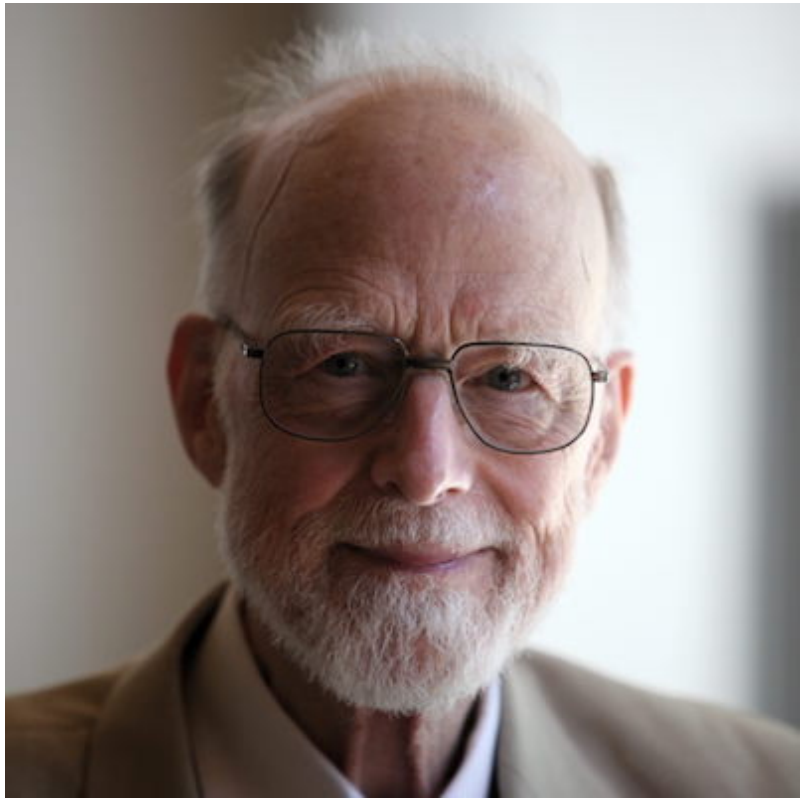
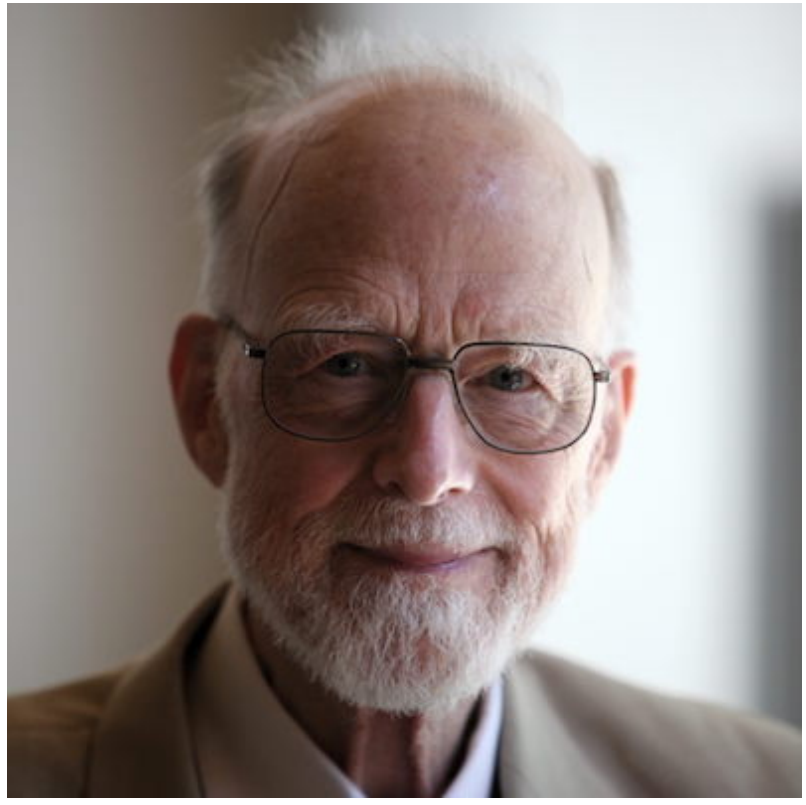
If $\models \{P\}\, c\, \{Q\}$ then $\vdash \{P\}\, c\, \{Q\}$.

vs.



Kurt Gödel

vs.



Sir Tony Hoare

# Relative Completeness

## Theorem (Cook (1974))

$\forall P, Q \in \mathbf{Assn}, c \in \mathbf{Com}.\ \vDash \{P\}\, c\, \{Q\}$ *implies* $\vdash \{P\}\, c\, \{Q\}$.

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## Theorem (Cook (1974))

$\forall P, Q \in \textbf{Assn}, c \in \textbf{Com}.\ \vDash \{P\}\, c\, \{Q\}$ *implies* $\vdash \{P\}\, c\, \{Q\}$.

## Proof Sketch.

Let $\{P\}\, c\, \{Q\}$ be a valid partial correctness specification.

By the first Lemma we have $\vDash P \implies \mathit{wlp}(c, Q)$.

By the second Lemma we have $\vdash \{\mathit{wlp}(c, Q)\}\, c\, \{Q\}$.

We conclude $\vdash \{P\}\, c\, \{Q\}$ using the Consequence rule. □