CS 4110

Programming Languages & Logics

Lecture 10
Denotational Semantics Proofs

Kleene Fixed-Point Theorem

Definition (Scott Continuity)

A function F is *Scott-continuous* if for every chain $X_1 \subseteq X_2 \subseteq \ldots$ we have $F(\bigcup_i X_i) = \bigcup_i F(X_i)$.

Kleene Fixed-Point Theorem

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A function F is *Scott-continuous* if for every chain $X_1 \subseteq X_2 \subseteq \ldots$ we have $F(\bigcup_i X_i) = \bigcup_i F(X_i)$.

Theorem (Kleene Fixed Point)

Let F be a Scott-continuous function. The least fixed point of F is $\bigcup_i F^i(\emptyset)$.

Denotational Semantics for IMP Commands

$$\mathcal{C}[\![\textbf{skip}]\!] = \{(\sigma, \sigma)\}$$

$$\mathcal{C}[\![x := a]\!] = \{(\sigma, \sigma[x \mapsto n]) \mid (\sigma, n) \in \mathcal{A}[\![a]\!]\}$$

$$\begin{aligned}\mathcal{C}[\![c_1; c_2]\!] = \\ &\{(\sigma, \sigma') \mid \exists \sigma''.\ ((\sigma, \sigma'') \in \mathcal{C}[\![c_1]\!] \wedge (\sigma'', \sigma') \in \mathcal{C}[\![c_2]\!])\}\end{aligned}$$

$$\begin{aligned}\mathcal{C}[\![\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2]\!] = \\ &\{(\sigma, \sigma') \mid (\sigma, \textbf{true}) \in \mathcal{B}[\![b]\!] \wedge (\sigma, \sigma') \in \mathcal{C}[\![c_1]\!]\} \ \cup \\ &\{(\sigma, \sigma') \mid (\sigma, \textbf{false}) \in \mathcal{B}[\![b]\!] \wedge (\sigma, \sigma') \in \mathcal{C}[\![c_2]\!]\}\end{aligned}$$

$$\begin{aligned}&\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!] = \mathit{fix}(F) \\ &\text{where } F(f) = \{(\sigma, \sigma) \mid (\sigma, \textbf{false}) \in \mathcal{B}[\![b]\!]\} \ \cup \\ &\qquad \{(\sigma, \sigma') \mid (\sigma, \textbf{true}) \in \mathcal{B}[\![b]\!] \wedge \exists \sigma''.\ ((\sigma, \sigma'') \in \mathcal{C}[\![c]\!] \wedge \\ &\qquad\qquad (\sigma'', \sigma') \in f\}\end{aligned}$$

Exercises

$\mathbf{skip}; c$ and $c; \mathbf{skip}$ are equivalent.

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skip; c and c; **skip** are equivalent.

$\mathcal{C}[\![\textbf{while false do } c]\!]$ is equivalent to...

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$\mathcal{C}[\![\textbf{while false do } c]\!]$ is equivalent to $\textbf{skip}$.

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$\mathbf{skip}; c$ and $c; \mathbf{skip}$ are equivalent.

$\mathcal{C}[\![\mathbf{while\ false\ do}\ c]\!]$ is equivalent to $\mathbf{skip}$.

$\mathcal{C}[\![\mathbf{while\ true\ do\ skip}]\!] = ?$