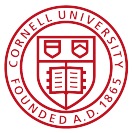
CS 4110 – Programming Languages and Logics
Lecture #19: More λ-calculus



1 Lambda calculus evaluation

There are many different evaluation strategies for the λ-calculus. The most permissive is *full β reduction*, which allows any *redex*—i.e., any expression of the form $(\lambda x.\ e_1)\ e_2$—to step to $e_1\{e_2/x\}$ at any time. It is defined formally by the following small-step operational semantics rules:

$$\frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2} \qquad \frac{e_2 \to e_2'}{e_1\ e_2 \to e_1\ e_2'} \qquad \frac{e_1 \to e_1'}{\lambda x.\, e_1 \to \lambda x.\, e_1'} \qquad \beta\ \frac{}{(\lambda x.\, e_1)\ e_2 \to e_1\{e_2/x\}}$$

The *call by value* (CBV) strategy enforces a more restrictive strategy: it only allows an application to reduce after its argument has been reduced to a value (i.e., a λ-abstraction) and does not allow evaluation under a λ. It is described by the following small-step operational semantics rules (here we show a left-to-right version of CBV):

$$\frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2} \qquad \frac{e_2 \to e_2'}{v_1\ e_2 \to v_1\ e_2'} \qquad \beta\ \frac{}{(\lambda x.\, e_1)\ v_2 \to e_1\{v_2/x\}}$$

Finally, the *call by name* (CBN) strategy allows an application to reduce even when its argument is not a value but does not allow evaluation under a λ. It is described by the following small-step operational semantics rules:

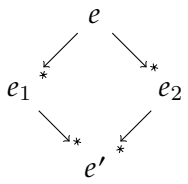
$$\frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2} \qquad \beta\ \frac{}{(\lambda x.\, e_1)\ e_2 \to e_1\{e_2/x\}}$$

2 Confluence

It is not hard to see that the full β reduction strategy is non-deterministic. This raises an interesting question: does the choices made during the evaluation of an expression affect the final result? The answer turns out to be no: full β reduction is *confluent* in the following sense:

Theorem (Confluence). *If $e \to^* e_1$ and $e \to^* e_2$ then there exists e' such that $e_1 \to^* e'$ and $e_2 \to^* e'$.*

Confluence can be depicted graphically as follows:



Confluence is often also called the Church–Rosser property.

3 Substitution

Each of the evaluation relations for λ-calculus has a β defined in terms of a substitution operation on expressions. Because the expressions involved in the substitution may share some variable names (and because we are working up to α-equivalence) the definition of this operation is slightly subtle and defining it precisely turns out to be tricker than might first appear.

As a first attempt, consider an obvious (but incorrect) definition of the substitution operator. Here we are substituting e for x in some other expression:

$$\begin{array}{rcll} y\{e/x\} & = & \begin{cases} e & \text{if } y = x \\ y & \text{otherwise} \end{cases} & \\ (e_1\ e_2)\{e/x\} & = & (e_1\{e/x\})\ (e_2\{e/x\}) & \\ (\lambda y.e_1)\{e/x\} & = & \lambda y.e_1\{e/x\} & \text{where } y \neq x \end{array}$$

The intuitive idea is that the last rule relies on α-equivalence to "rewrite" abstractions that use x so they do not conflict. Unfortunately, this definition produces the wrong results when we substitute an expression with free variables under a λ. For example,

$$(\lambda y.x)\{y/x\} = (\lambda y.y)$$

To fix this problem, we need to revise our definition so that when we substitute under a λ we do not accidentally bind variables in the expression we are substituting. The following definition correctly implements *capture-avoiding substitution*:

$$\begin{array}{rcll} y\{e/x\} & = & \begin{cases} e & \text{if } y = x \\ y & \text{otherwise} \end{cases} & \\ (e_1\ e_2)\{e/x\} & = & (e_1\{e/x\})\ (e_2\{e/x\}) & \\ (\lambda y.e_1)\{e/x\} & = & \lambda y.(e_1\{e/x\}) & \text{where } y \neq x \text{ and } y \notin fv(e) \end{array}$$

Note that in the case for λ-abstractions, we require that the bound variable y be different from the variable x we are substituting for and that y not appear in the free variables of e, the expression we are substituting. Because we work up to α-equivalence, we can always pick y to satisfy these side conditions. For example, to calculate $(\lambda z.x\ z)\{(w\ y\ z)/x\}$ we first rewrite $\lambda z.x\ z$ to $\lambda u.x\ u$ and then apply the substitution, obtaining $\lambda u.(w\ y\ z)\ u$ as the result.

4 λ-calculus encodings

The pure λ-calculus contains only functions as values. It is not exactly easy to write large or interesting programs in the pure λ-calculus. We can however encode objects, such as booleans, and integers.

4.1 Booleans

Let us start by encoding constants and operators for booleans. That is, we want to define functions TRUE, FALSE, AND, NOT, IF, and other operators that behave as expected. For example:

$$\begin{aligned} \mathsf{AND}\ \mathsf{TRUE}\ \mathsf{FALSE} &= \mathsf{FALSE} \\ \mathsf{NOT}\ \mathsf{FALSE} &= \mathsf{TRUE} \\ \mathsf{IF}\ \mathsf{TRUE}\ e_1\ e_2 &= e_1 \\ \mathsf{IF}\ \mathsf{FALSE}\ e_1\ e_2 &= e_2 \end{aligned}$$

Let's start by defining TRUE and FALSE:

$$\begin{aligned} \mathsf{TRUE} &\triangleq \lambda x.\, \lambda y.\, x \\ \mathsf{FALSE} &\triangleq \lambda x.\, \lambda y.\, y \end{aligned}$$

Thus, both TRUE and FALSE are functions that take two arguments; TRUE returns the first, and FALSE returns the second. We want the function IF to behave like

$$\lambda b.\, \lambda t.\, \lambda f.\, \text{if } b = \mathsf{TRUE} \text{ then } t \text{ else } f.$$

The definitions for TRUE and FALSE make this very easy.

$$\mathsf{IF} \triangleq \lambda b.\, \lambda t.\, \lambda f.\, b\ t\ f$$

Definitions of other operators are also straightforward.

$$\begin{aligned} \mathsf{NOT} &\triangleq \lambda b.\, b\ \mathsf{FALSE}\ \mathsf{TRUE} \\ \mathsf{AND} &\triangleq \lambda b_1.\, \lambda b_2.\, b_1\ b_2\ \mathsf{FALSE} \\ \mathsf{OR} &\triangleq \lambda b_1.\, \lambda b_2.\, b_1\ \mathsf{TRUE}\ b_2 \end{aligned}$$

4.2 Church numerals

Church numerals encode a number n as a function that takes f and x, and applies f to x n times.

$$\begin{aligned} \overline{0} &\triangleq \lambda f.\, \lambda x.\, x \\ \overline{1} &= \lambda f.\, \lambda x.\, f\ x \\ \overline{2} &= \lambda f.\, \lambda x.\, f\ (f\ x) \\ \mathsf{SUCC} &\triangleq \lambda n.\, \lambda f.\, \lambda x.\, f\ (n\ f\ x) \end{aligned}$$

In the definition for SUCC, the expression $n\ f\ x$ applies f to x n times (assuming that variable n is the Church encoding of the natural number n). We then apply f to the result, meaning that we apply f to x $n + 1$ times.

Given the definition of SUCC, we can easily define addition. Intuitively, the natural number $n_1 + n_2$ is the result of apply the successor function n_1 times to n_2.

$$\mathsf{PLUS} \triangleq \lambda n_1.\, \lambda n_2.\, n_1\ \mathsf{SUCC}\ n_2$$