



We have now seen two operational models for programming languages: small-step and large-step. In this lecture, we consider a different semantic model, called *denotational semantics*.

The idea in denotational semantics is to express the meaning of a program as the mathematical function that expresses what the program computes. We can think of an IMP program c as a function from stores to stores: given an initial store, the program produces a final store. For example, the program $\text{foo} := \text{bar} + 1$ can be thought of as a function that when given an input store σ , produces a final store σ' that is identical to σ except that it maps foo to the integer $\sigma(\text{bar}) + 1$; that is, $\sigma' = \sigma[\text{foo} \mapsto \sigma(\text{bar}) + 1]$. We will model programs as functions from input stores to output stores. As opposed to operational models, which tell us *how* programs execute, the denotational model shows us *what* programs compute.

1 A Denotational Semantics for IMP

For each program c , we write $C[[c]]$ for the *denotation* of c , that is, the mathematical function that c represents:

$$C[[c]] : \text{Store} \rightarrow \text{Store}.$$

Note that $C[[c]]$ is actually a partial function (as opposed to a total function), both because the store may not be defined on the free variables of the program and because program may not terminate for certain input stores. The function $C[[c]]$ is not defined for non-terminating programs as they have no corresponding output stores.

We will write $C[[c]]\sigma$ for the result of applying the function $C[[c]]$ to the store σ . That is, if f is the function that $C[[c]]$ denotes, then we write $C[[c]]\sigma$ to mean the same thing as $f(\sigma)$.

We must also model expressions as functions, this time from stores to the values they represent. We will write $\mathcal{A}[[a]]$ for the denotation of arithmetic expression a , and $\mathcal{B}[[b]]$ for the denotation of boolean expression b .

$$\mathcal{A}[[a]] : \text{Store} \rightarrow \text{Int}$$

$$\mathcal{B}[[b]] : \text{Store} \rightarrow \{\text{true}, \text{false}\}$$

Now we want to define these functions. To make it easier to write down these definitions, we will describe (partial) functions using sets of pairs. More precisely, we will represent a partial map $f : A \rightarrow B$ as a set of pairs $F = \{(a, b) \mid a \in A \text{ and } b = f(a) \in B\}$ such that, for each $a \in A$, there is at most one pair of the form $(a, _)$ in the set. Hence $(a, b) \in F$ is the same as $b = f(a)$.

We can now define denotations for IMP. We start with the denotations of expressions:

$$\begin{aligned}\mathcal{A}[\![n]\!] &= \{(\sigma, n)\} \\ \mathcal{A}[\![x]\!] &= \{(\sigma, \sigma(x))\} \\ \mathcal{A}[\![a_1 + a_2]\!] &= \{(\sigma, n) \mid (\sigma, n_1) \in \mathcal{A}[\![a_1]\!] \wedge (\sigma, n_2) \in \mathcal{A}[\![a_2]\!] \wedge n = n_1 + n_2\}\end{aligned}$$

$$\begin{aligned}\mathcal{B}[\![\mathbf{true}]\!] &= \{(\sigma, \mathbf{true})\} \\ \mathcal{B}[\![\mathbf{false}]\!] &= \{(\sigma, \mathbf{false})\} \\ \mathcal{B}[\![a_1 < a_2]\!] &= \{(\sigma, \mathbf{true}) \mid (\sigma, n_1) \in \mathcal{A}[\![a_1]\!] \wedge (\sigma, n_2) \in \mathcal{A}[\![a_2]\!] \wedge n_1 < n_2\} \cup \\ &\quad \{(\sigma, \mathbf{false}) \mid (\sigma, n_1) \in \mathcal{A}[\![a_1]\!] \wedge (\sigma, n_2) \in \mathcal{A}[\![a_2]\!] \wedge n_1 \geq n_2\}\end{aligned}$$

The denotations for commands are as follows:

$$\begin{aligned}C[\![\mathbf{skip}]\!] &= \{(\sigma, \sigma)\} \\ C[\![x := a]\!] &= \{(\sigma, \sigma[x \mapsto n]) \mid (\sigma, n) \in \mathcal{A}[\![a]\!]\} \\ C[\![c_1; c_2]\!] &= \{(\sigma, \sigma') \mid \exists \sigma''. ((\sigma, \sigma'') \in C[\![c_1]\!] \wedge (\sigma'', \sigma') \in C[\![c_2]\!])\}\end{aligned}$$

Note that $C[\![c_1; c_2]\!] = C[\![c_2]\!] \circ C[\![c_1]\!]$, where $\circ$ is the composition of relations, defined as follows: if $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$ then $R_2 \circ R_1 \subseteq A \times C$ is $R_2 \circ R_1 = \{(a, c) \mid \exists b \in B. (a, b) \in R_1 \wedge (b, c) \in R_2\}$. If $C[\![c_1]\!]$ and $C[\![c_2]\!]$ are total functions, then $\circ$ is function composition.

$$\begin{aligned}C[\![\mathbf{if } b \mathbf{ then } c_1 \mathbf{ else } c_2]\!] &= \{(\sigma, \sigma') \mid (\sigma, \mathbf{true}) \in \mathcal{B}[\![b]\!] \wedge (\sigma, \sigma') \in C[\![c_1]\!]\} \cup \\ &\quad \{(\sigma, \sigma') \mid (\sigma, \mathbf{false}) \in \mathcal{B}[\![b]\!] \wedge (\sigma, \sigma') \in C[\![c_2]\!]\} \\ C[\![\mathbf{while } b \mathbf{ do } c]\!] &= \{(\sigma, \sigma) \mid (\sigma, \mathbf{false}) \in \mathcal{B}[\![b]\!]\} \cup \\ &\quad \{(\sigma, \sigma') \mid (\sigma, \mathbf{true}) \in \mathcal{B}[\![b]\!] \wedge \exists \sigma''. ((\sigma, \sigma'') \in C[\![c]\!] \wedge (\sigma'', \sigma') \in C[\![\mathbf{while } b \mathbf{ do } c]\!])\}\end{aligned}$$

But now we have a problem: the last “definition” is not really a definition because it expresses $C[\![\mathbf{while } b \mathbf{ do } c]\!]$ in terms of itself! This is not a definition but a recursive equation. What we want is the solution to this equation.

2 Fixed Points

We gave a recursive equation that the function $C[\![\mathbf{while } b \mathbf{ do } c]\!]$ must satisfy. To understand some of the issues involved, let’s consider a simpler example. Consider the following equation for a function $f : \mathbb{N} \rightarrow \mathbb{N}$.

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x-1) + 2x - 1 & \text{otherwise} \end{cases} \quad (1)$$

This is not a definition for f , but rather an equation that we want f to satisfy. What function, or functions, satisfy this equation for f ? The only solution to this equation is the function $f(x) = x^2$.

In general, there may be no solutions for a recursive equation (e.g., there are no functions $g : \mathbb{N} \rightarrow \mathbb{N}$ that satisfy the recursive equation $g(x) = g(x) + 1$), or multiple solutions (e.g., find two functions $g : \mathbb{R} \rightarrow \mathbb{R}$ that satisfy $g(x) = 4 \times g(\frac{x}{2})$).

We can compute solutions to such equations by building successive approximations. Each approximation is closer and closer to the solution. To solve the recursive equation for f , we start with the partial function $f_0 = \emptyset$ (i.e., f_0 is the empty relation; it is a partial function with the empty set for its domain). We compute successive approximations using the recursive equation.

$$\begin{aligned}
f_0 &= \emptyset \\
f_1 &= \begin{cases} 0 & \text{if } x = 0 \\ f_0(x-1) + 2x - 1 & \text{otherwise} \end{cases} \\
&= \{(0, 0)\} \\
f_2 &= \begin{cases} 0 & \text{if } x = 0 \\ f_1(x-1) + 2x - 1 & \text{otherwise} \end{cases} \\
&= \{(0, 0), (1, 1)\} \\
f_3 &= \begin{cases} 0 & \text{if } x = 0 \\ f_2(x-1) + 2x - 1 & \text{otherwise} \end{cases} \\
&= \{(0, 0), (1, 1), (2, 4)\} \\
&\vdots
\end{aligned}$$

This sequence of successive approximations f_i gradually builds the function $f(x) = x^2$.

We can model this process of successive approximations using a higher-order function F that takes one approximation f_k and returns the next approximation f_{k+1} :

$$F : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow (\mathbb{N} \rightarrow \mathbb{N})$$

where

$$(F(f))(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x-1) + 2x - 1 & \text{otherwise} \end{cases}$$

A solution to the recursive equation [1](#) is a function f such that $f = F(f)$. In general, given a function $F : A \rightarrow A$, we have that $a \in A$ is a *fixed point* of F if $F(a) = a$. We also write $a = \text{fix}(F)$ to indicate that a is a least fixed point of F .

So the solution to the recursive equation [1](#) is a fixed-point of the higher-order function F . We can compute this fixed point iteratively, starting with $f_0 = \emptyset$ and at each iteration computing $f_{k+1} = F(f_k)$. The fixed point is the limit of this process:

$$\begin{aligned}
f &= \text{fix}(F) \\
&= f_0 \cup f_1 \cup f_2 \cup f_3 \cup \dots \\
&= \emptyset \cup F(\emptyset) \cup F(F(\emptyset)) \cup F(F(F(\emptyset))) \cup \dots \\
&= \bigcup_{i \geq 0} F^i(\emptyset)
\end{aligned}$$

3 Denotation for Loops

We can now write the correct denotation case for **while** loops as the fixed point of a higher-order function:

$$\begin{aligned} C \llbracket \mathbf{while} \ b \ \mathbf{do} \ c \rrbracket &= \text{fix}(F) \\ \text{where } F(f) &= \{(\sigma, \sigma) \mid (\sigma, \mathbf{false}) \in \mathcal{B} \llbracket b \rrbracket\} \cup \\ &\quad \{(\sigma, \sigma') \mid (\sigma, \mathbf{true}) \in \mathcal{B} \llbracket b \rrbracket \wedge \exists \sigma''. (\sigma, \sigma'') \in C \llbracket c \rrbracket \wedge (\sigma'', \sigma') \in f\} \end{aligned}$$