CS 4110
Lab #24 – Lambda Calculus

Recall the Church numerals

$\bar{0} = \lambda s. \lambda z. z$
$\bar{1} = \lambda s. \lambda z. s\ z$
$\bar{2} = \lambda s. \lambda z. s\ (s\ z)$

<u>Exercise</u>: Define ISEVEN and DOUBLE.

$\text{ISEVEN} \triangleq \lambda n.\ n\ \underbrace{(\lambda b.\ b\ \text{FALSE}\ \text{TRUE})}_{\text{NOT}}\ \text{TRUE}$

$\text{DOUBLE} \triangleq \lambda n. \lambda s. \lambda z.\ n\ s\ (n\ s\ z)$

<u>Exercise</u>: Define PAIR, FST, SND

$\text{PAIR} \triangleq \lambda x. \lambda y. \lambda b.\ b\ x\ y$
$\text{FST} \triangleq \lambda p.\ p\ \text{TRUE}$
$\text{SND} \triangleq \lambda p.\ p\ \text{FALSE}$

<u>Exercise</u>: Define PRED (with $\text{PRED}\ \bar{0} =_\beta \bar{0}$.)

$\text{PRED} \triangleq \lambda n.\ \text{SND}\ (n\ (\lambda p.\ \text{PAIR}\ (\text{SUCC}\ (\text{FST}\ p))\ (\text{FST}\ p))$
$\quad (\text{PAIR}\ \bar{0}\ \bar{0}))$

Here is another version that doesn't use pairs (due to Barendregt)

$$\text{PRED} \triangleq \lambda n.\lambda s.\lambda z.\ n\ \overbrace{(\lambda p.\lambda q.\ \underset{\uparrow\ \text{continuation}}{q}\ (\underset{\uparrow\ \text{function so far}}{p}\ s))}^{\text{BUILD}}\ \overbrace{(\lambda x.z)}^{\text{END}\ \cancel{\text{START}}}\ \overbrace{(\lambda x.x)}^{\text{START}}$$

$$\text{PRED}\ \bar{2}$$

$$= \text{PRED}\ \lambda s.\lambda z.\ s\ (s\ z)$$

$$=_\beta \lambda s.\lambda z.\ (\lambda p.\lambda q.\ q\ (p\ s))\ ((\lambda p.\lambda q.\ q\ (p\ s))\ \lambda x.z)\ (\lambda x.x)$$

$$=_\beta \lambda s.\lambda z\ \ (\lambda p.\lambda q.\ q\ (p\ s))\ (\lambda q.\ q\ ((\lambda x.z)\ s))\ (\lambda x.x)$$

$$=_\beta \lambda s.\lambda z.\ (\lambda q.\ q\ ((\lambda q.\ q\ (\lambda x.z)\ s)\ s))\ (\lambda x.x)$$

$$=_\beta \lambda s.\lambda z.\ (\lambda q.\ q\ (s\ z))\ (\lambda x.x)$$

$$=_\beta \lambda s.\lambda z\ s\ z$$

Exercise: Define EQUAL

$$\text{EQUAL} = \lambda n.\lambda m.\ \text{AND}\ (\text{ISZERO}\ (m\ \text{PRED}\ n))\ (\text{ISZERO}\ (n\ \text{PRED}\ m))$$

where

$$\text{AND} = \lambda b.\lambda c.\ b\ c\ \text{FALSE}$$

Scott numerals

$\hat{0} = \lambda s.\lambda z.\ z$
$\hat{1} = \lambda s.\lambda z.\ s\ \hat{0}$
$\hat{2} = \lambda s.\lambda z.\ s\ \hat{1}$

Intuition: case analysis!
- successor (of n-1)
- zero

Define: SUCC, PRED, ISZERO

$\text{succ} = \lambda n.\lambda s.\lambda z.\ s\ n$
$\text{pred} = \lambda n.\ n\ (\lambda p.p)\ \hat{0}$
$\text{iszero} = \lambda n.\ n\ (\lambda p.\ \text{FALSE})\ \text{TRUE}$

Define: ADD

$\text{ADD} = \lambda n.\lambda m.\ n\ \ \text{SUCC}\ (m\ p)$

Define: C2S and S2C

$\text{C2S} = \lambda n.\ n\ \ \text{SUCC}_s\ \hat{0}$
$\text{S2C} = \lambda n.\ n\ \ \text{SUCC}_c\ \overline{0}$