CS 4110 - 9/3/25
Lab #4 - Semantics

I. Progress Proof

Lemma [Progress]

$$\forall e \in Exp\ \forall \sigma \in Store.\ \underbrace{wf(\sigma, e) \Rightarrow \overbrace{e \in Int \vee \exists \sigma' \in Store, e' \in Exp.\ \langle \sigma, e\rangle \to \langle \sigma', e'\rangle}^{C(e,\sigma)}}_{P(e)}$$

Proof. By induction on (the structure of) e.

Case $e = x$.

Let $\sigma \in$ Store and assume $wf(\sigma, x)$.

By defn of wf, we have $fvs(e) \subseteq dom(\sigma)$.

So, as $fvs(x) = \{x\}$, we have $\exists n \in Int.\ \sigma(x) = n$.

By **Var**, we have $\langle \sigma, x\rangle \to \langle \sigma, n\rangle$, which finishes the case.

Case $e = e_1 + e_2$.

Let $\sigma \in$ Store and assume $wf(\sigma, e_1 + e_2)$

By defn of wf, we have $fvs(e_1 + e_2) \subseteq dom(\sigma)$.

So, as $fvs(e_1 + e_2) = fvs(e_1) \cup fvs(e_2)$, we have $fvs(e_1) \subseteq dom(\sigma)$ and $fvs(e_2) \subseteq dom(\sigma)$.

$\therefore$ $wf(\sigma, e_1)$ and $wf(\sigma, e_2)$.

By IH on e_1, we have $C(e_1, \sigma)$. We analyze both sub-cases

Sub-case $e_1 \in Int$:

So $e_1 = n_1$.

By IH on e_2, we have $C(e_2, \sigma)$. Again, we analyze both sub-sub-cases.

Sub-sub-case $e_2 \in \mathrm{Int}$:

So $e_2 = n_2$.

Let $n = n_1 + n_2$.

By Add, we have $\langle \sigma, n_1 + n_2 \rangle \to \langle \sigma, n \rangle$, which finishes the sub-sub-case.

Sub-sub-case $\exists \sigma', e'.\ \langle \sigma, e_2 \rangle \to \langle \sigma', e' \rangle$:

By RAdd, we have $\langle \sigma, n_1 + e_2 \rangle \to \langle \sigma', n_1 + e' \rangle$, which finishes the sub-sub-case, and the sub-case.

Sub-case $\exists \sigma', e'.\ \langle \sigma, e_1 \rangle \to \langle \sigma', e' \rangle$:

By LAdd, we have $\langle \sigma, e_1 + e_2 \rangle \to \langle \sigma', e' + e_2 \rangle$, which finishes the sub-case, and the case!

II. Mini ML Semantics

Consider this language

$$
\begin{aligned}
e ::= {} & n \\
\mid {} & e_1 + e_2 \\
\mid {} & [\,] \\
\mid {} & e_1 :: e_2 \\
\mid {} & e_1 @ e_2
\end{aligned}
\qquad\qquad
\begin{aligned}
l ::= {} & [\,] \\
\mid {} & n :: l
\end{aligned}
$$

Notation

$[n_1, n_2, n_3 \ldots]$ means $n_1 :: n_2 :: n_3 :: \ldots :: [\,]$

Exercise: give small-step rules for $e_1 :: e_2$ and $e_1 @ e_2$.

$$
\frac{e_1 \to e_1'}{e_1 :: e_2 \to e_1' :: e_2}\ \text{CL}
\qquad
\frac{e_2 \to e_2'}{n_1 :: e_2 \to n_1 :: e_2'}\ \text{CR}
$$

$$\frac{e_1 \to e_1'}{e_1 @ e_2 \to e_1' @ e_2}\ \text{AL} \qquad \frac{e_2 \to e_2'}{l_1 @ e_2 \to l_1 @ e_2'}\ \text{AR}$$

$$\frac{}{[\,] @ l_2 \to l_2}\ \text{AN} \qquad \frac{}{(n_1 :: l_1) @ l_2 \to n_1 :: (l_1 @ l_2)}\ \text{AC}$$

Exercise: give the series of steps for $[1,2] @ [3,4]$.

$[1,2] @ [3,4]$

$\to 1 :: ([2] @ [3,4])$ By AC

$\to 1 :: 2 :: ([\,] @ [3,4])$ By CR, AC

$\to 1 :: 2 :: [3,4]$ By CR, CR, AN

$= [1,2,3,4]$ Notation

Now let's extend the language

$e ::= \ldots$

$\quad \mid \text{rev } e$

$\quad \mid \text{len } e$

$$\frac{e \to e'}{\text{rev } e \to \text{rev } e'}\ \text{RE} \qquad \frac{}{\text{rev } [\,] \to [\,]}\ \text{RN} \qquad \frac{}{\text{rev } (n::\ell) \to (\text{rev } \ell) @ (n::[\,])}\ \text{RC}$$

$$\frac{e \to e'}{\text{len } e \to \text{len } e'}\ \text{RE} \qquad \frac{}{\text{len } [\,] \to 0}\ \text{LN} \qquad \frac{}{\text{len } (n::\ell) \to 1 + \text{len } \ell}\ \text{LC}$$

Now let's extend it one more time

$$\begin{aligned} e ::= &\ \dots \\ &\mid x \\ &\mid \text{match } e \text{ with } [\,] \to e_1 \mid x::xs \to e_2 \end{aligned}$$

$$\frac{e \to e'}{\text{match } e \text{ with } [\,] \to e_1 \mid x::xs \to e_2 \to \text{match } e' \text{ with } [\,] \to e_1 \mid x::xs \to e_2}\ \text{ME}$$

$$\frac{}{\text{match } [\,] \text{ with } [\,] \to e_1 \mid x::xs \to e_2 \to e_1}\ \text{MN}$$

$$\frac{}{\text{match } n::\ell \text{ with } [\,] \to e_1 \mid x::xs \to e_2 \to e_2[n/x, \ell/xs]}$$

substitution!

Bonus: Can you define hd and tl?